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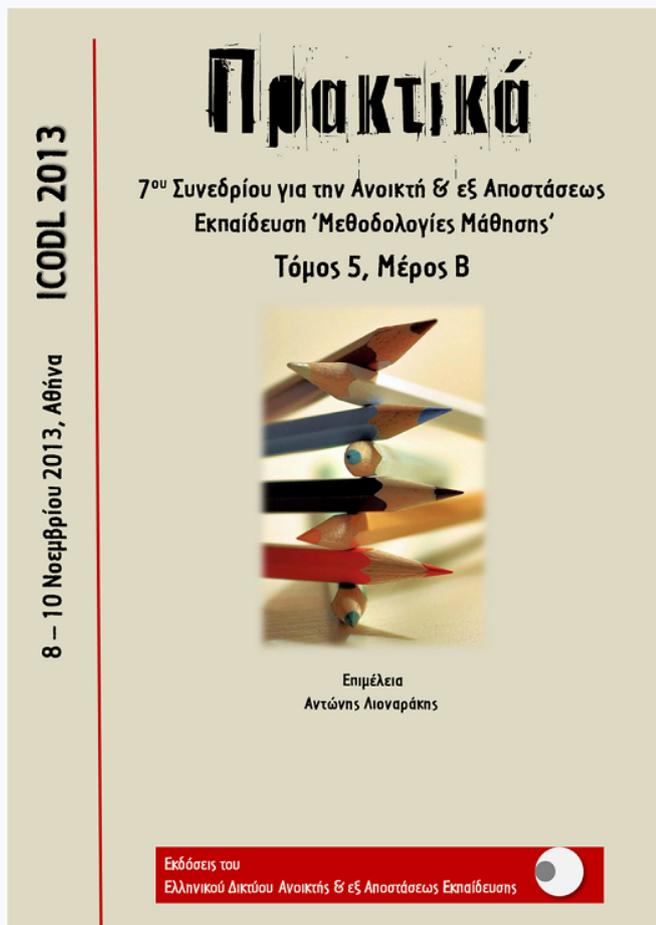
Students' realization of the role of contour curves for functions from R2 to R

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Students' realization of the role of contour curves for functions from R^2 to R

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Abstract
The present study consists of three components: a) analysis of the mental images that students construct in their minds through the use of contour curves in order to visualize the graph of a function of two variables in $\square^3$, b) investigation of the conceptions of students of how to relate contour curves with some of the methods used to find the limit of a function of two variables, so that they may arrive at a conclusion regarding the existence of the limit of a function in $\square^2$ and c) explanation of the type of difficulties that students face and the dynamic transformations that they use through the construction of contour curves during their effort to visualize the graph of a function of two variables in polar coordinate system. To examine the afore-mentioned components, a field study was implemented with the participation of 75 students. The experimental group, which attended a several variables Analysis course, consisted of second year students in the Department of Mathematics of University of Patras. For data collection, a test on both the thematic axes was set, and interviews with five students were employed. The analysis focuses on ways in which students use the mental images they create to solve the test tasks. Results that can support prospective studies on the role of contour curves are presented.

Key-words: *contour curves, limit, function of two-variables, visualization*

Introduction

In recent years, the tertiary educational institutes that teach Mathematics have been placing more and more emphasis on the use of visualization and graphical representations in the teaching of Mathematics, in order to assist students in clarifying and better consolidating various mathematical concepts, and also in arriving at various conclusions about them. The field of *visualization*, or *spatial thinking,* is very broad and includes numerous terms such as mental images, spatial images, imagination etc.

In accordance with Kosslyn (1980), the term *visualization* means the kind of reasoning based on the use of mental images. A *mental image* is a mental representation of a mathematical concept or property containing information based on pictorial, symbolic, graphical or diagrammatic elements. One or many mental images may consist of the *surface representation*, the quasi-pictorial entity present in the active memory, and *a deep representation*, the information stored in the long-term memory from which the surface representation is derived.

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A *spatial image* derives from the sensory cognition of spatial relations, and can be expressed in a range of verbal or graphical forms (Yakimanskaya, 1991).

The major technological developments made in recent years have enabled students to transform different objects dynamically by selection through the use of a wide-ranging spectrum of computing systems. The aim of this study is to analyze students' mental images formed through external representations, such as the contour curves constructed by students themselves using a pencil in order to illustrate the graph of a two-variable function, and to draw conclusions. In a subsequent research study, the aim will be to use dynamic software to enable students to make various transformations, such as rotations, translations, enlargements or plane sections, and investigate their abilities to do so when working in such an environment. (Figure 1 describes the steps that students follow when using visualization to solve a task.)
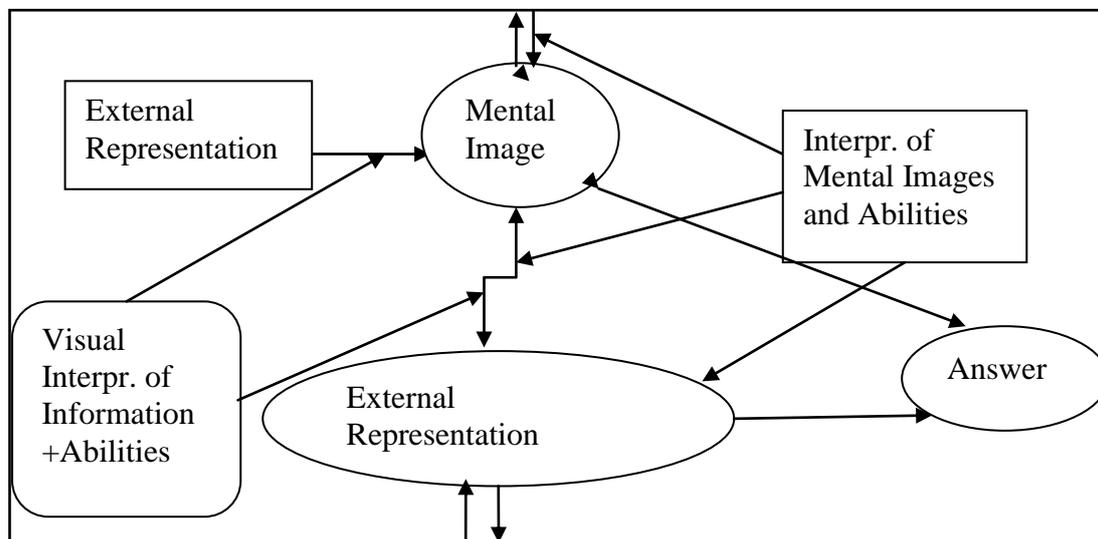
Task

Fig. 1. (Gutiérrez, 1996)

Consequently, within this framework, the research questions of this study are the following:

- What is the reasoning that can be developed or employed by students in order to analyze visual indications (e.g. contour curves) in order to construct the graph of a function of two variables? Can students foresee something different from this that the tutor expects?
- What are the difficulties that students may face in the visualization and conceptual understanding of a function of two variables?
- What are the conclusions drawn (or reasoning employed) by students about the limit of this function through the construction of contour curves of a two-variable function or its graph in three-dimensional space?
- What are the resulting misunderstandings and dynamic images when students are asked to construct the contour curves of a function of two variables or to visualize the graph of this function in another coordinate system such as the polar coordinate system?

Methodology

The context of the present study is a several variables Analysis course, part of the mathematics education program taught at a faculty of the Department of Mathematics at a university in Southern Greece. The study was conducted with the participation of 75 students, whose tutor was the author of the study, during the fall semester of the academic year 2012-2013. The context of the course on functions of two variables included finding of the domain and range, the methods used for finding their limits, the teaching of contour curves and drawings of some special surfaces. A test was then prepared and given to 75 second-year mathematics students of average competence. The duration of the test was two hours.

Five students were chosen by the tutor and separately called for interview (each was videotaped and lasted for approximately 50 minutes) in order to describe and substantiate their test answers. Specifically, the in-depth interview was used (Boyce & Neale, 2006). The selection of the students for interview was based upon two criteria: firstly, these cases offered rich and comprehensive information able to provide answers to the research questions and, secondly, their collective work well represented all of the behavior observed. The type of interview structure selected was the semi-structured interview (Silverman, 2000).

From the very first teaching session of the course, students were informed that a test would follow at the end of the sessions and that their performance in the test would not affect their final evaluation. Moreover, it was pointed out that their participation in the test would be voluntary and completion of their personal data would be optional (Weinberg, 2002).

Additionally, with regard to the interview process, students were informed about the exact duration of the interview. Although it is true that the author of this paper had the roles of tutor, researcher, interviewer and observer, under no circumstances did she assume that of evaluator, aiming to create a climate of trust and comfort between her and the students throughout the duration of the fieldwork and the interview, in order to collect as much data as possible for her research. Finally, the interviewer tried to be as neutral and objective in the interview as possible (Cohent & Manion, 1992) and not overrate or underrate the truth that each interviewee professes (Cannel & Kann, 1968).

Tasks and rationale in designing them

Task 1: a. Find the domain of the function $z = f(x, y) = 100 - x^2 - y^2$.

b. Sketch the contour curves $z = c$, for $c = 0, c = 51, c = 75$ and $c = 100$. Describe the contour curves.

c. Estimate the limit of the function $f(x, y)$ when $(x, y) \rightarrow (0,0)$,

$(x, y) \rightarrow (5,0)$, $(x, y) \rightarrow (0,7)$ and $(x, y) \rightarrow (5\sqrt{2}, 5\sqrt{2})$.

What do you observe?

d. Sketch the graph of the surface $f(x, y) = 100 - x^2 - y^2$?

In (b), students need to construct the contour curves of *f*. Task 1 is a classic example that we meet in most Real Analysis books. However, by asking students to find the limits (c), through the process of converting algebraic representations into geometric representations (b), the goal is for them to make an initial conjecture about there being some connection between the contour curves and the limit of a two-variable function,

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which they are to verify and substantiate in Task 2. The goal in (d) is to verify if students are able to correlate their dynamic images for Cartesian plane, contour curves and their intuitive notion of three-dimensional space through actions (b). Namely, (d) investigates if students' constructions of the contour curves facilitate their visualization of the graph of f in space $\square^3$.

Task 2: Consider the function $f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & ,(x, y) \neq (0,0) \\ 0 & ,(x, y) = (0,0) \end{cases}$.

    a. Sketch the contour curves for $c = -1, 0$ and 1.
    b. Investigate if the $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exists by using:
    i. directional approach and
    ii. sequence definition of convergence.
    c. Would you arrive at the same conclusion with regard to the existence of the limit of f at $(0,0)$ from question (a)? Justify your answer.
    d. How do you imagine the form of the graph of f in $\square^3$?

Directional approach is the approach when the domain of f is restricted to half-lines, lines (or more generally curves) for finding the limits of functions from $\square^2$ to $\square$ (Mamona-Downs & Megalou, 2013).

The sequence definition of convergence, states that:

Given $U \subset \square^2$, $f : U \subset \square$, $(x_0, y_0) \in \square^2$ accumulation point of U and $l \in \square$, then

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = l$ if for every sequence $(x_n, y_n) \subset U - \{(x_0, y_0)\}$:

$(x_n, y_n) \to (x_0, y_0) \Rightarrow f(x_n, y_n) \to l$.

Consequently, according to the afore- mentioned methods used for finding the limit of a function of two real variables at a point (x_0, y_0), this Task investigates how students can correlate contour curves (a), the directional approach and the sequence definition of convergence (b, c) in order to draw a conclusion about the existence of the limit of function $f(x, y)$ at a point (x_0, y_0). Moreover, question (d) in this Task investigates the mental constructions of students in order to determine the graph of $f(x, y)$ in $\square^3$.

Task 3:

In the polar coordinate system (r, ϑ) the function $f(r, \vartheta) = \begin{cases} \sin 2\vartheta, & if\,(r, \vartheta) \neq (0,0) \\ 0, & if\,(r, \vartheta) = (0,0) \end{cases}$

is given, where $(0,0)$ is the origin (point of reference) of the system.

    a. Sketch the contour curves $f(r, \vartheta) = c$, for $c = \pm 1/2, \pm 1$, respectively.
    b. How do you imagine the form of the graph of f in $\square^3$?

This is a two-level Task: Firstly, it investigates whether students are able to construct contour curves in the polar coordinate system, and secondly it examines the process through which students depict the graph of $f(x, y)$ in $\square^3$.

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Results

Some of the most characteristic and noteworthy answers given by the five students resulted from the test in which they participated, but some extracts from their interviews are also mentioned and briefly commented upon below. Wherever their answers are similar, these are not mentioned. Students are given the letters T, N, L, E, D, the interviewer with letter I.

Task 1 (d): In Task 1(d), T had drawn a sphere. In his interview, he stated:

Contour curves are obtained by intersections of the graph of $z = f(x,y)$ by the plane $z = c$, parallel to xy-plane. As c obtains greater values, r decreases and subsequently the contour degenerates to a point. The sphere has the same property.

Later, reviewing the algebraic form $z = f(x, y)$ he reconsiders his answer, saying that, since contour curves are circles the graph of *f* would resemble an inverted bell. A similar image to that of T was drawn by N, who drew a half-sphere. She drew a circular paraboloid because, as she stated, she remembered from the Analytical Geometry course she has taken that the equation represents a circular paraboloid.

Finally, E, ignorant of the basic role of contour curves, could not draw the graph of *f*, although she did properly draw the contour curves of *f*.

In Task 2 b (i, ii), 4 out of 5 students applied the directional approach and the sequence definition of convergence properly and easily. Their answers differed in question c. Subsequently, we cite the incorrect answer of student L:

Suppose that we have two sequences $x_n = 1/n \to 0$ and $y_n = 1/n \to 0$, $\forall n \in \square$, then the corresponding sequence of the values of *f* is $f(x_n, y_n) = 2\dfrac{(1/n)(1/n)}{1/n^2 + 1/n^2} = 1$.

Thus, $\lim\limits_{(x_n, y_n) \to (0,0)} f(x_n, y_n) = 1 \neq 0$,

and from this he believes that he has enough information to prove that the limit of *f* does not exist.

This answer by L also influenced him in the answer he gave to Task 2c, where he wrote:

Yes, indeed, the limit of *f* does not exist; this is shown by the contour curves of *f*, since the values of *f* vary at the point. The limits to the right differ from those to the left.

In fact, in his interview, he stated that:

We have two straight lines passing through $(0,0)$, that is, as if we have two sequences, $x_n = 1/n \to 0$ and $y_n = 1/n \to 0$, that give "different images", i.e. -1 and 1.

L would appear to understand (from the construction of contour curves) that the limit of *f* does not exist, but the analytic representation of his answer shows that he confuses the concept of sequence in $\square$ and $\square^2$. However, from the answer given by L, a combination of contour curves, directional approach and sequence definition of convergence would appear to be lurking in his mind.

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Next, we quote the answer of N:

Indeed, it is possible to arrive at the same conclusion with regard to the existence of the limit of f at $(0,0)$ using only contour curves. This is because the contour curves intersect at $(0,0)$. In fact, N gives an additional example of a surface $z = 1 - x - y$ citing that:

The contour curves of f are straight lines passing through $(0,0)$ and, therefore, the limit of f exists.

However, in her interview, she is unable to rationalize why this occurs. Finally, T arrives at the same conclusion as that of N by citing the example of Task 2a and surface $z = y^2 / (x^2 + y^2)$.

Task 2 (d): With regard to Task 2(d), in his interview L said that:

In order for contour curves to be straight lines, and by observing the algebraic form, I see that the surface will certainly have parabolas. Of course, the xy factor worries me. Perhaps there is a gap there. It is as if I have a piece of paper and pull it down at its centre and, ultimately, form a 'mountain range'.

Similar mental constructions were also expressed by T, who stated:

Contour curves depict the shade of a three-dimensional shape. Consequently, in order for them to become straight lines that intersect at a point, I believe that the graph of f would resemble a paper ship; namely, a piece of paper that is crumpled up in the middle.

N drew two planes running through each other and, in her interview, stated that:

As contour curves are all that remain in the xy - plane, if I cut the shape, or otherwise view what shape is formed from above, I believe that this would be the graph of f.

Finally, D visualized the graph of f as two inverted funnels with a small gap between them, giving a similar reason to that of L, while E, although she sketched the contour curves, had difficulty in conceiving how the graph of this function would be.

Task 3: L was the only one of the five students who stated that contour curves of f are different pairs of straight lines resulting for the various values of $z = c$. In Task 3(b), L answered that:

For the value $c = 1/2$, I found the angle $\vartheta = \pi/12$. I sketched this angle in the polar system and subsequently brought the parallel from the point $(\pi/2, 1/2)$ to the horizontal axis. Then I found the point of intersection with the side of angle $\vartheta = \pi/12$. Continuing the same process for the appropriate values for c, I found the corresponding points on the contour curves (straight lines) and I observed that, if I join them together, a petal is formed. Seeing the imprint formed in the xy- plane, I imagine that the graph of f would probably have the form of an apple in $\square^3$.

E created the same shape as that of L. However, she was of the opinion that the contour curves of f are the four petals and not the straight lines formed by the petals, as shown in the following extract from the interview:

1. I: Why do you believe that the contour curves of f are the four petals formed?

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2. E: I took the graphical representation of the sine curve and for $0 \le \vartheta \le \pi/2$ I transferred the part I could see to the polar axis. Continuing the same process, I noticed that a flower was formed.

Additionally, due to an algebraic error, D formed two petals symmetrical to axis $\pi/2$ and 0 and insisted that the contour curves are the lemniscates formed. As he stated in his interview, D imagined the graph of f as a bow. T came up with a similar solution to that of E and visualized the graph of f as a pepper in three-dimensional space. Although N found the values of angle ϑ for the various values of c, she terminated the process at that point.

Discussion and conclusions

The results of the present study demonstrated under interview students encountered several difficulties in the visualization of $f(x, y)$ in $\square^3$. In particular, one student (L) tried to recall a shape that he associated with a particular set of symbols. He focused on connecting a symbolic representation he had associated with a particular shape, stating that: 'if I remember well, from the Analytical Geometry course, the equation represents a circular paraboloid'. Therefore, it appeared that he was describing an already generated graph of a function of two variables, rather than imagining its construction from contour perspectives. Moreover, two students (E, D) could not visualize the graph of f in $\square^3$, although they did properly draw the contour curves of f. It would appear that these students had a restricted understanding of the nature of contour curves, solely performing operational procedures with different c's. Two other students (T and N) appeared to have a sense of contour curves but a misunderstanding of the algebraic form of f led them to visualize the graph of f in $\square^3$ incorrectly. The difficulties that students faced with regard to the graphs of functions of two variables have also been recorded in other studies (Martinez-Planell, & Trigueros-Gaisman, 2012; Trigueros & Martinez-Planell, 2010; Kabael, 2009). As Picard argues (2010), these difficulties may be due to the poor ability of the majority of students to 'see' in three-dimensional space.

Moreover, the test given to the students also contained more difficult graphs to construct, but here the aim was to investigate how students could imagine the process in which a shape was constructed in space through contour curves. The findings of the study show that two students (L, D) associated a set of symbols with memorized images of graphical representations of Algebra in order to describe the graph of f in three-dimensional space. Moreover, the use of a phrase like "if I cut the shape" shows that a student visualizes and expresses an object common to his everyday existence rather than a *priori* mathematical object. Similar results have been found in the study by Trigueros- Gaisman and Martinez-Planell (2010), who believe that the potential of intersecting surfaces with planes, and predicting the result of this intersection, plays a fundamental role in understanding graphs of functions of two variables.

In general, the ability of students to carry out actions such as operations and procedures related to the construction of contour curves and the application of the two methods used for finding the limit of f in $\square^2$ correctly was observed. Only one of the five students (L) failed to mimic the structure and to relate the features of the sequence definition of convergence form in the function of one variable to those in the function of two variables. However, he did realize that there is some connection

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between contour curves and two other methods. Another two students (N, T) had the same perception, as they speculated that the limit of f does not exist by comparing it with the example of Task 2a or other similar examples without, however, being able to justify their result. Only one of them (T) seems to have given an indication of proper justification based on the sequence definition of convergence.

Finally, the static images that students created through the contour curves encouraged them to construct fine dynamic images for the graph of f in $\square^3$. However, there was confusion as to which of these are contours (only two said they are the formed lines), suggesting that students believed in a dependence between the two variables r, ϑ, which contradicts the definition of contour curves and consequently confirms the subjects' poor understanding of the concept of contour curves. The confusion of students with regard to the variables in the polar coordinate system is also evident in other studies, such as those of Montiel et al (2008), in which it was ascertained that students fail to convey the concept of one variable from the Cartesian coordinate system to the polar coordinate system because the majority of them consider variable r as independent and ϑ as dependent, rather than the opposite.

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