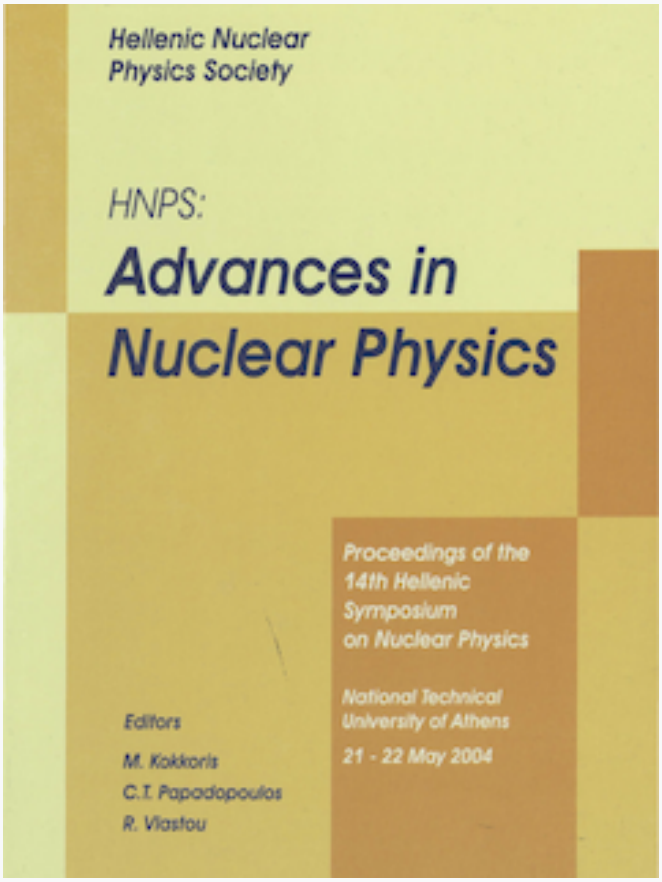
HNPS Advances in Nuclear Physics

Vol 13 (2004)

HNPS2004



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doi: 10.12681/hnps.2952

Variational procedure leading from Davidson potentials to the E(5)and X(5) critical point symmetries

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Abstract

Davidson potentials of the form $\beta^2 + \beta_0^4/\beta^2$, when used in the original Bohr Hamiltonian for γ-independent potentials bridge the U(5) and O(6) symmetries. Using a variational procedure, we determine for each value of angular momentum L the value of β_0 at which the derivative of the energy ratio $R_L = E(L)/E(2)$ with respect to β_0 has a sharp maximum, the collection of R_L values at these points forming a band which practically coincides with the ground state band of the E(5) model, corresponding to the critical point in the shape phase transition from U(5) to O(6). The same potentials, when used in the Bohr Hamiltonian after separating variables as in the X(5) model, bridge the U(5) and SU(3) symmetries, the same variational procedure leading to a band which practically coincides with the ground state band of the X(5) model, corresponding to the critical point of the U(5) to SU(3) shape phase transition. A new derivation of the Holmberg–Lipas formula for nuclear energy spectra is obtained as a by-product.

Key words: Davidson potential, Shape phase transition, Variational method, E(5) model, X(5) model
PACS: 21.60.Ev, 21.60.Fw

1 Introduction

The recently introduced E(5) [1] and X(5) [2] models are supposed to describe shape phase transitions in atomic nuclei, the former being related to

[1] Deceased

the transition from U(5) (vibrational) to O(6) (γ-unstable) nuclei, and the latter corresponding to the transition from U(5) to SU(3) (rotational) nuclei. In both cases the original Bohr collective Hamiltonian [3] is used, with an infinite well potential in the collective β-variable. Separation of variables is achieved in the E(5) case by assuming that the potential is independent of the collective γ-variable, while in the X(5) case the potential is assumed to be of the form $u(\beta)+u(\gamma)$. We are going to refer to these two cases as "the E(5) framework" and "the X(5) framework" respectively. In the present work we examine if the choice of the infinite well potential is the optimum one for the description of shape phase transitions. For this purpose, we need one-parameter potentials which can span the U(5)-O(6) region in the E(5) framework, as well as the U(5)-SU(3) region in the X(5) framework. It turns out that the exactly soluble [4,5] Davidson potentials [6]

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}, \tag{1}$$

where β_0 is the position of the minimum of the potential, do possess this property. Taking into account the fact that various physical quantities should change most rapidly at the point of the shape phase transition [7], we locate for each value of the angular momentum L the value of β_0 for which the rate of change of the ratio $R_L = E(L)/E(2)$, a widely used measure of nuclear collectivity, is maximized. It turns out that the collection of R_L ratios formed in this way in the case of a potential independent of the γ-variable correspond to the E(5) model, while in the case of the $u(\beta)+u(\gamma)$ potential lead to the X(5) model, thus proving that the choice of the infinite well potential made in Refs. [1,2] is the optimum one.

2 Davidson potentials in the E(5) framework

The original Bohr Hamiltonian [3] is

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}\right] + V(\beta,\gamma), \tag{2}$$

where β and γ are the usual collective coordinates describing the shape of the nuclear surface, Q_k ($k = 1, 2, 3$) are the components of angular momentum, and B is the mass parameter.

Assuming that the potential depends only on the variable β, i.e. $V(\beta,\gamma) = U(\beta)$, one can proceed to separation of variables in the standard way [3,8], using the wave function $\Psi(\beta,\gamma,\theta_i) = f(\beta)\Phi(\gamma,\theta_i)$, where θ_i $(i = 1,2,3)$ are the Euler angles describing the orientation of the deformed nucleus in space. In the equation involving the angles, the eigenvalues of the second order Casimir operator of SO(5) occur, having the form $\Lambda = \tau(\tau+3)$, where $\tau = 0, 1, 2, \ldots$ is the quantum number characterizing the irreducible representations (irreps) of SO(5), called the "seniority" [9]. This equation has been solved by Bes [10]. The "radial" equation can be simplified by introducing [1] reduced energies $\epsilon = \frac{2B}{\hbar^2}E$ and reduced potentials $u = \frac{2B}{\hbar^2}U$, leading to

$$\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{\tau(\tau+3)}{\beta^2} + u(\beta)\right] f(\beta) = \epsilon f(\beta). \tag{3}$$

When plugging the Davidson potentials of Eq. (1) in the above equation, the β_0^4/β^2 term is combined with the $\tau(\tau+3)/\beta^2$ term appearing there and the equation is solved exactly [4,5], the eigenfunctions being Laguerre polynomials, while the energy eigenvalues are given by [4,5] (in $\hbar\omega = 1$ units)

$$E_{n,\tau} = 2n + 1 + \left[\left(\tau + \frac{3}{2}\right)^2 + \beta_0^4\right]^{1/2}. \tag{4}$$

The levels of the ground state band are characterized by $L = 2\tau$ and $n = 0$. Then the energy levels of the ground state band are given by

$$E_{0,L} = 1 + \frac{1}{2}\left[(L+3)^2 + 4\beta_0^4\right]^{1/2}, \tag{5}$$

while the excitation energies of the levels of the ground state band relative to the ground state are

$$E_{0,L,exc} = E_{0,L} - E_{0,0} = \frac{1}{2}\left(\left[(L+3)^2 + 4\beta_0^4\right]^{1/2} - \left[9 + 4\beta_0^4\right]^{1/2}\right). \tag{6}$$

For $u(\beta)$ being a 5-D infinite well one obtains the E(5) model of Iachello [1] in which the eigenfunctions are Bessel functions $J_{\tau+3/2}(z)$ (with $z = \beta k$, $k = \sqrt{\epsilon}$), while the spectrum is determined by the zeros of the Bessel functions.

It is instructive to consider the ratios

$$R_L = \frac{E_{0,L} - E_{0,0}}{E_{0,2} - E_{0,0}}, \tag{7}$$

where the notation $E_{n,L}$ is used. For $\beta_0 = 0$ it is clear that the original vibrational model of Bohr [3] (with $R_4 = 2$) is obtained, while for large β_0 the O(6)

Table 1
Parameter values $\beta_{0,max}$ where the first derivative of the energy ratios R_L (defined in Eq. (7)) in the E(5) framework has a maximum, while the second derivative vanishes, together with the R_L ratios obtained at these values (labeled by "var") and the corresponding ratios of the E(5) model, for several values of the angular momentum L.

L	$\beta_{0,max}$	R_L var	R_L E(5)
4	1.421	2.185	2.199
6	1.522	3.549	3.590
8	1.609	5.086	5.169
10	1.687	6.793	6.934
12	1.759	8.667	8.881
14	1.825	10.705	11.009
16	1.888	12.906	13.316
18	1.947	15.269	15.799
20	2.004	17.793	18.459

limit of the Interacting Boson Model (IBM) [12] (with $R_4 = 2.5$) is approached [4]. One can easily see that these ratios increase with β_0, the increase becoming very steep at some value $\beta_{0,max}$ of β_0, where the first derivative $\frac{dR_L}{d\beta_0}$ reaches a maximum value, while the second derivative $\frac{d^2R_L}{d\beta_0^2}$ vanishes. Then the values of β_0 at which the first derivative $dR_L/d\beta_0$ exhibits a sharp maximum are determined for each value of the angular momentum L separately, the collection of R_L ratios at these values of β_0 forming a band, which turns out to be in very good agreement with the ground state band of E(5), the model supposed to be appropriate for describing nuclei at the critical point in the transition from U(5) to O(6), thus indicating that the choice of the infinite well potential used in the E(5) model is the optimum one. The results are depicted in Table 1.

3 Davidson potentials in the X(5) framework

Starting again from the original Bohr Hamiltonian [3], one seeks solutions of the relevant Schrödinger equation of the form $\Psi(\beta, \gamma, \theta_i) = \phi_K^L(\beta, \gamma) D_{M,K}^L(\theta_i)$, where θ_i (i = 1, 2, 3) are the Euler angles, $D(\theta_i)$ denote Wigner functions of them, L are the eigenvalues of angular momentum, while M and K are the eigenvalues of the projections of angular momentum on the laboratory-fixed z-axis and the body-fixed z'-axis respectively. As pointed out in Ref. [2], an approximate separation of variables can be achieved by assuming that

the potential has a minimum around $\gamma = 0$, as well as that it can be separated into two terms, one depending on β and the other depending on γ, i.e. $u(\beta, \gamma) = u(\beta) + u(\gamma)$. As described in more detail in Ref. [11], the "radial" equation occuring in this case can be solved exactly in the case of the Davidson potentials of Eq. (1), the energy eigenvalues being (in $\hbar\omega = 1$ units)

$$E_{n,L} = 2n + 1 + \left[\frac{1}{3}L(L+1) + \frac{9}{4} + \beta_0^4\right]^{1/2} \tag{8}$$

The levels of the ground state band are characterized by $n = 0$. With appropriate normalization we get the expression

$$E'_{0,L,exc} = \frac{E_{0,L,exc}}{\left[\frac{9}{4} + \beta_0^4\right]^{1/2}} = \left[1 + \frac{L(L+1)}{3\left(\frac{9}{4} + \beta_0^4\right)}\right]^{1/2} - 1, \tag{9}$$

which is the same as the Holmberg–Lipas formula [13]

$$E_H(L) = a_H\left(\sqrt{1 + b_H L(L+1)} - 1\right), \tag{10}$$

with $a_H = 1$ and $b_H = 1/(27/4 + 3\beta_0^4)$.

It is clear that the Holmberg–Lipas formula gives rotational spectra for small values of b_H, at which one can keep only the first L-dependent term in the Taylor expansion of the square root appearing in Eq. (10), leading to energies proportional to $L(L+1)$. From the expression for b_H it is then clear that rotational spectra are expected for large values of β_0. On the other hand, the case $\beta_0 = 0$ corresponds to an exactly soluble model with $R_4 = 2.646$, which has been called the X(5)-β^2 model [14]. Following the same variational procedure as before, we produce a collection of R_L ratios in very good agreement with the ground state band of the X(5) model, as is shown in Table 2.

4 Discussion

A variational procedure for determining the values of physical quantities at the point of shape phase transitions in nuclei has been suggested. Using one-parameter potentials spanning the region between the two limiting symmetries of interest, the parameter values at which the rate of change of the physical quantity becomes maximum are determined for each value of the angular momentum separately and the corresponding values of the physical quantity at these parameter values are calculated.

Table 2
Parameter values $\beta_{0,max}$ where the first derivative of the energy ratios R_L (defined in Eq. (7)) in the X(5) framework has a maximum, while the second derivative vanishes, together with the R_L ratios obtained at these values (labeled by "var") and the corresponding ratios of the X(5) model, for several values of the angular momentum L.

L	$\beta_{0,max}$	R_L var	R_L X(5)
4	1.334	2.901	2.904
6	1.445	5.419	5.430
8	1.543	8.454	8.483
10	1.631	11.964	12.027
12	1.711	15.926	16.041
14	1.785	20.330	20.514
16	1.855	25.170	25.437
18	1.922	30.442	30.804
20	1.985	36.146	36.611

The method has been applied in the shape phase transition from U(5) to O(6), using one-parameter Davidson potentials [6] and considering the energy ratios $R_L = E(L)/E(2)$ within the ground state band as the relevant physical quantity, leading to a band which practically coincides with the ground state band of the E(5) model [1]. It has also been applied in the same way in the shape phase transition from U(5) to SU(3), leading to a band which practically coincides with the ground state band of the X(5) model [2].

It should be emphasized that the application of the method was possible because the Davidson potentials correctly reproduce the U(5) and O(6) symmetries in the former case (for small and large parameter values respectively), as well as the relevant X(5)-β^2 [14] and SU(3) symmetries in the latter case (for small and large parameter values respectively).

As a by-product, a derivation of the Holmberg–Lipas formula [13] has been achieved using Davidson potentials in the X(5) framework.

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