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SYMMETRIES IN ODD-MASS NUCLEI

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Abstract: A general review is given of applications of algebraic techniques to odd-mass nuclei in the framework of the interacting boson-fermion model. The discussion focuses on the symmetry aspect of the model: the concepts of symmetry and dynamical symmetry are illustrated by way of elementary examples. The unified treatment of even-even and odd-mass nuclei (supersymmetry) is introduced by inspecting the action of 'super'generators creating a boson and annihilating a fermion (or *vice versa*). These ideas are illustrated with some examples. The talk is concludes with a brief summary.

1. Introduction

The interacting boson model has emerged in the last fifteen years as a unified framework for the description of collective properties of nuclei. The key ingredients of this model are its algebraic structure based on the powerful methods of group theory, the possibility it gives to perform calculations in all but few nuclei and its direct connection with the shell model that allows one to derive its properties from microscopic interactions. A vast literature exists discussing the results and implications of the interacting boson model. It consists of numerous articles and several review papers, and also two monographs [1,2] have been written on the subject.

The interacting boson model deals with nuclei with an even number of protons and neutrons. However, more than half of the nuclear species have an odd number of protons and/or neutrons. In these nuclei there is an interplay between collective (bosonic) and single-particle (fermionic) degrees of freedom. The interacting boson model was extended to cover these situations by introducing the interacting boson-fermion model [3] which was subsequently expanded by Iachello and Scholten [4] and cast into a form more readily amenable to calculations.

No attempt is made here to present a comprehensive review of the interacting boson–fermion model. Rather, it is my intention to analyze the model from one specific viewpoint, namely its algebraic structure. Other aspects—such as its geometric interpretation (i.e., connection with the Nilsson model) or its microscopic structure (connection with the shell model)—are not discussed at all or only peripherally mentioned. The reader interested in further details may find these in [5].

I shall start the discussion with a brief outline of the essential idea of the interacting boson–fermion model.

2. The interacting boson–fermion model

In the interacting boson–fermion model the collective degrees of freedom are described by bosons operators. To lowest order of approximation only bosons with angular momentum and parity $J^\pi = 0^+$ and 2^+ are retained (s- and d-bosons). The corresponding creation and annihilation operators are written as

$$b^\dagger_{lm}, \quad b_{lm}, \qquad l = 0, 2, \quad -l \leq m \leq l \tag{1}$$

and satisfy the commutation relations

$$[b_{lm}, b^\dagger_{l'm'}] = \delta_{ll'}\delta_{mm'}, \qquad [b_{lm}, b_{l'm'}] = [b^\dagger_{lm}, b^\dagger_{l'm'}] = 0 \tag{2}$$

In addition to collective degrees of freedom one wants to describe single-particle degrees of freedom. In nuclei the single particles are neutrons and protons. These are fermions. The angular momentum of these particles depends on the allowed orbits and it is denoted by j with z-component m. The parity can be either positive or negative. An interacting boson–fermion model is specified by the number and the values of angular momenta retained. In treating the single-particle degrees of freedom, it is also convenient to use the formalism of second quantization and introduce the fermion creation and annihilation operators

$$a^\dagger_{jm}, \quad a_{jm}, \qquad m = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \ldots, \pm j \tag{3}$$

These operators satisfy anticommutation relations

$$\{a_{jm}, a^\dagger_{j'm'}\} = \delta_{jj'}\delta_{mm'}, \qquad \{a_{jm}, a_{j'm'}\} = \{a^\dagger_{jm}, a^\dagger_{j'm'}\} = 0 \tag{4}$$

Finally, it is assumed that boson and fermion operators commute

$$[b_{lm}, a_{j'm'}] = [b_{lm}, a^\dagger_{j'm'}] = [b^\dagger_{lm}, a_{j'm'}] = [b^\dagger_{lm}, a^\dagger_{j'm'}] = 0 \tag{5}$$

This is a natural assumption if bosons and fermions are elementary particles. In nuclei, where bosons are composite particles (fermion pairs), it is a model assumption. Effects of the compositeness of the bosons are introduced through additional interactions.

All operators that correspond to physical observables must now be written in terms of the boson and fermion operators introduced so far. I illustrate the procedure with the example of the hamiltonian operator; it contains a boson part $\hat{H}_{\rm B}$, a fermion part $\hat{H}_{\rm F}$ and a boson–fermion interaction $\hat{V}_{\rm BF}$,

$$\hat{H} = \hat{H}_{\rm B} + \hat{H}_{\rm F} + \hat{V}_{\rm BF} \tag{6}$$

It is assumed that the hamiltonian conserves separately the number of bosons, $N_{\rm B}$, and the number of fermions, $N_{\rm F}$. The structure of the various parts of the hamiltonian operator then reduces to

$$\begin{aligned}
\hat{H}_{\rm B} &= E_0 + \sum_l \epsilon_l \sqrt{2l+1}[b^\dagger_l \times \tilde{b}_l]^{(0)}_0 \\
&+ \sum_{L_{\rm B}, ll'l''l'''} \tfrac{1}{2} u^{(L_{\rm B})}_{ll'l''l'''} [[b^\dagger_l \times b^\dagger_{l'}]^{(L_{\rm B})} \times [\tilde{b}_{l''} \times \tilde{b}_{l'''}]^{(L_{\rm B})}]^{(0)}_0 + \cdots \\
\hat{H}_{\rm F} &= \mathcal{E}_0 - \sum_j \eta_j \sqrt{2j+1}[a^\dagger_j \times \tilde{a}_j]^{(0)}_0 \\
&+ \sum_{L_{\rm F}, jj'j''j'''} \tfrac{1}{2} v^{(L_{\rm F})}_{jj'j''j'''} [[a^\dagger_j \times a^\dagger_{j'}]^{(L_{\rm F})} \times [\tilde{a}_{j''} \times \tilde{a}_{j'''}]^{(L_{\rm F})}]^{(0)}_0 + \cdots \\
\hat{V}_{\rm BF} &= -\sum_{J, ljl'j'} w^{(J)}_{ljl'j'} \sqrt{2J+1}[[b^\dagger_l \times a^\dagger_j]^{(J)} \times [\tilde{b}_{l'} \times \tilde{a}_{j'}]^{(J)}]^{(0)}_0 + \cdots
\end{aligned} \tag{7}$$

where ϵ_l, $u^{(L_{\rm B})}_{ll'l''l'''}$, η_j, $v^{(L_{\rm F})}_{jj'j''j'''}$ and $w^{(J)}_{ljl'j'}$ are parameters of the model. Note that the expressions (7) are expansions of which only the first few terms are given. The coupling to zero angular momentum of the various terms in (7) ensures that the model is rotationally invariant and that the hamiltonian has eigenstates with good angular momentum.

The calculational procedure now appears straightforward. It consists of diagonalizing the hamiltonian (7) in a basis with N bosons $b^\dagger_{lm}$ (N is the number of nucleon pairs in the valence shell) and a single fermion $a^\dagger_{jm}$. This produces an eigenspectrum corresponding to an odd-mass nucleus. The resulting eigenvectors can then be used to compute further properties of this nucleus (e.g., electromagnetic transition rates). The approach followed in the interacting boson–fermion model is very much like the standard particle–core coupling models [6] but with the core described in terms of a set of interacting bosons. The analysis in the interacting boson–fermion model is, however, considerably simplified because of its algebraic structure and the presence of (dynamical) symmetries. Before turning to that topic, I will make a few remarks about symmetries and dynamical symmetries in general.

3. Symmetries and dynamical symmetries

I begin with some elementary considerations related to *symmetry*, mainly to introduce the notation used in this section. For applications in low-energy nuclear physics the idea of symmetry is most conveniently introduced via a hamiltonian formalism. A hamiltonian $\hat{H}$, invariant under a set of (infinitesimal) transformations $\hat{g}_i$ which together form a Lie algebra G, i.e.

$$[\hat{H}, \hat{g}_i] = 0 \qquad \text{for } \hat{g}_i \in G \tag{8}$$

is said to have a symmetry G or, alternatively, to be invariant under G. A well-known consequence of a symmetry is the occurrence of degeneracies in the eigenspectrum of $\hat{H}$. Given an eigenstate ψ of $\hat{H}$ with energy E, the condition (8) implies that the states $\hat{g}_i\psi$ have the same energy. This enables one to write an arbitrary eigenstate of $\hat{H}$ as $|\Gamma\gamma\rangle$, where the first quantum number Γ is different for states with different energies and the second quantum number γ is needed to distinguish degenerate eigenstates. The energy eigenvalues of a hamiltonian satisfying (8) thus depend only on Γ,

$$\hat{H}|\Gamma\gamma\rangle = E(\Gamma)|\Gamma\gamma\rangle \tag{9}$$

and, furthermore, the transformations $\hat{g}_i$ do not admix states with different Γ:

$$\hat{g}_i|\Gamma\gamma\rangle = \sum_{\gamma'} a^{i,\Gamma}_{\gamma,\gamma'}|\Gamma\gamma'\rangle \tag{10}$$

In the language of group theory the transformation coefficients $a^{i,\Gamma}_{\gamma,\gamma'}$, when considered as matrices in the indices γ and γ', provide a (matrix) *representation* of the elements $\hat{g}_i$ in the vector space spanned by the states $|\Gamma\gamma\rangle$. The representation obviously depends on Γ and is denoted as $[\Gamma]$. Another ingredient borrowed from group theory concerns the construction of operators like $\hat{H}$ in (8) that commute with all elements of G. Such operators are called *Casimir operators* and are denoted here as $C_n[G]$, the index n referring to the order of the operator in the $\hat{g}_i$. The enumeration of independent Casimir operators associated with a given algebra G and the derivation of expressions for their eigenvalues $E_n(\Gamma)$,

$$C_n[G]|\Gamma\gamma\rangle = E_n(\Gamma)|\Gamma\gamma\rangle \tag{11}$$

is a well-studied problem, the solution of which can be found in many monographs on group theory (see, e.g., [7,8]).

Next I introduce the concept of *dynamical symmetry*,[1] for which one needs (at least) two algebras G_1 and G_2 with $G_1 \supset G_2$. First the condition of G_1 symmetry is imposed on the hamiltonian $\hat{H}$ and, as before, its eigenstates can be labelled as $|\Gamma_1\gamma_1\rangle$. But, since $G_1 \supset G_2$, a hamiltonian with G_1 symmetry necessarily must also have a symmetry G_2 and, consequently, its eigenstates can also be labelled as $|\Gamma_2\gamma_2\rangle$. Combination of the two properties leads to the eigenequation

$$\hat{H}|\Gamma_1\Gamma_2\gamma_2\rangle = E(\Gamma_1)|\Gamma_1\Gamma_2\gamma_2\rangle \tag{12}$$

where the role of γ_1 is played by $\Gamma_2\gamma_2$ and hence the eigenvalues depend only on Γ_1.[2] The meaning of the labels used in (12) is further illustrated with the transformation properties of the states $|\Gamma_1\Gamma_2\gamma_2\rangle$ under the action of an element belonging to G_1 or G_2:

$$\begin{aligned}
\hat{g}_i|\Gamma_1\Gamma_2\gamma_2\rangle &= \sum_{\Gamma_2'\gamma_2'} a^{i,\Gamma_1}_{\Gamma_2\gamma_2,\Gamma_2'\gamma_2'}|\Gamma_1\Gamma_2'\gamma_2'\rangle \qquad \text{for } \hat{g}_i \in G_1 \\
\hat{g}_i|\Gamma_1\Gamma_2\gamma_2\rangle &= \sum_{\gamma_2'} a^{i,\Gamma_2}_{\gamma_2,\gamma_2'}|\Gamma_1\Gamma_2\gamma_2'\rangle \qquad \text{for } \hat{g}_i \in G_2
\end{aligned} \tag{13}$$

[1] I follow the nomenclature and conventions adopted in the interacting boson model.

[2] In (12) I have excluded the possibility that the same representation $[\Gamma_2]$ occurs more than once in $[\Gamma_1]$, in which case one would need an additional quantum number α to uniquely label the states as $|\Gamma_1\alpha\Gamma_2\gamma_2\rangle$. For the purpose of illustrating the concept of dynamical symmetry, however, one may ignore this technical complication.

In many applications the condition of G_1 symmetry is found to be too strong and must be relaxed. A *possible* breaking of the G_1 symmetry occurs via the hamiltonian

$$\hat{H}' = a\,C_{n_1}[G_1] + b\,C_{n_2}[G_2] \tag{14}$$

The essential idea is to take a combination of Casimir operators of G_1 *and* G_2. Let us now look at the symmetry properties of the hamiltonian $\hat{H}'$. Since $[\hat{H}', \hat{g}_i] = 0$ for $\hat{g}_i \in G_2$, $\hat{H}'$ is invariant under G_2. However, the hamiltonian $\hat{H}'$ in general does not commute with all elements of G_1 and for this reason the G_1 symmetry is broken, the extent of the symmetry breaking depending on the ratio b/a. Furthermore, since $\hat{H}'$ is written as a combination of Casimir operators of G_1 and G_2, its eigenvalues are obtained in closed form:

$$\Big(a\,C_{n_1}[G_1] + b\,C_{n_2}[G_2]\Big)|\Gamma_1\Gamma_2\gamma_2\rangle = \Big(aE_{n_1}(\Gamma_1) + bE_{n_2}(\Gamma_2)\Big)|\Gamma_1\Gamma_2\gamma_2\rangle. \tag{15}$$

Thus we conclude that, although $\hat{H}'$ is not invariant under G_1, its eigenstates are the same as those of $\hat{H}$ in (12). The hamiltonian $\hat{H}'$ is said to have G_1 as a dynamical symmetry. The essential feature is that, although the eigenvalues of $\hat{H}'$ depend on Γ_1 *and* Γ_2 (and hence G_1 is not a symmetry), the eigenstates do not change during the breaking of the G_1 symmetry: the dynamical symmetry breaking splits but does not admix the eigenstates.

These ideas can be illustrated with some well-known examples. The first is taken from nuclear physics and concerns isospin multiplets of nuclei [9]. To describe a system of interacting neutrons and protons we might, in first approximation, assume the hamiltonian to be isospin invariant, since that is a symmetry property which we believe to be valid for the strong interaction. In the notation introduced above, G_1 in this example is the isospin algebra $SU_T(2)$, consisting of the operators $\hat{T}_+$, $\hat{T}_z$ and $\hat{T}_-$, and G_2 should be identified with $O_T(2) = \{\hat{T}_z\}$, the projection operator on the z-axis in isospin space. An isospin-invariant hamiltonian commutes with $\hat{T}_+$, $\hat{T}_z$ and $\hat{T}_-$, and hence the eigenstates $|TM_T\rangle$ with fixed T and $M_T = -T, -T+1, \ldots, +T$ are degenerate in energy. (Compare with (9) by making the substitutions $\Gamma \to T$ and $\gamma \to M_T$.) Unlike the strong interaction the electromagnetic interaction is not isospin invariant and will lift the degeneracy of the states $|TM_T\rangle$. We assume that this symmetry breaking occurs dynamically and, furthermore, it can be shown that, since the Coulomb force has a two-body character, the breaking term is at most quadratic in $\hat{T}_z$ [10]. Under these two restrictions the energies of corresponding nuclear states belonging to the same isospin multiplet are given by

$$E(M_{\mathrm{T}}) = a + bM_{\mathrm{T}} + cM_{\mathrm{T}}^2 \tag{16}$$

Two conclusions are obtained from these considerations. First, because the electromagnetic symmetry breaking is assumed to occur in a *dynamical* manner, eigenstates of the nuclear hamiltonian have good T and M_{T}. Second, the two-body character of the Coulomb force leads to the expansion (16) for the energies of corresponding nuclear states with the same T. This formula can be tested for a $T = 3/2$ multiplet consisting of isobaric analog states in ^{13}B, ^{13}C, ^{13}N and ^{13}O. In figure 1 are shown the binding energies of the nuclei ^{13}B and ^{13}O, both of which have $T = |M_{\mathrm{T}}| = 3/2$ in their ground state. The isobaric analog states in ^{13}C and ^{13}N are $J^\pi = 3/2^-$ states at excitation energies of 15.11 and 15.07 MeV, respectively; these energies are substracted from the binding energies of ^{13}C and ^{13}N to give the energies plotted in figure 1. In this example the energy splitting due to the Coulomb force clearly is well accounted for by the energy formula (16), which is perhaps not surprising since four data points are fitted with three parameters. However, it should be emphasized that the quality of fits such as in figure 1 is not the most important aspect of dynamical symmetries, but rather the existence of good quantum numbers (T and M_{T} in this case).

The next example is taken from particle physics and concerns the classification of 'elementary' particles into SU(3) multiplets [11]. In this case the relevant symmetry algebras and their associated quantum numbers are

$$\begin{array}{ccccccc} \mathrm{SU}(3) & \supset & \mathrm{U_Y}(1) & \otimes & \big(\mathrm{SU_T}(2) & \supset & \mathrm{O_T}(2)\big) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\lambda,\mu) & & Y & & T & & M_{\mathrm{T}} \end{array} \tag{17}$$

where T and M_{T} are the isospin and its projection on the z-axis and Y is the hypercharge. Instead of the notation (λ,μ), which I follow here [12], SU(3) representations often are denoted by their dimension, that is, the number of independent basis vectors in the representation (= the number of particles in the corresponding SU(3) multiplet). If one assumes SU(3) invariance, all particles belonging to one multiplet are predicted to have the same mass. Since the observed masses differ by hundreds of MeV, clearly some SU(3)-symmetry breaking terms must be introduced. However, SU(3) can be broken

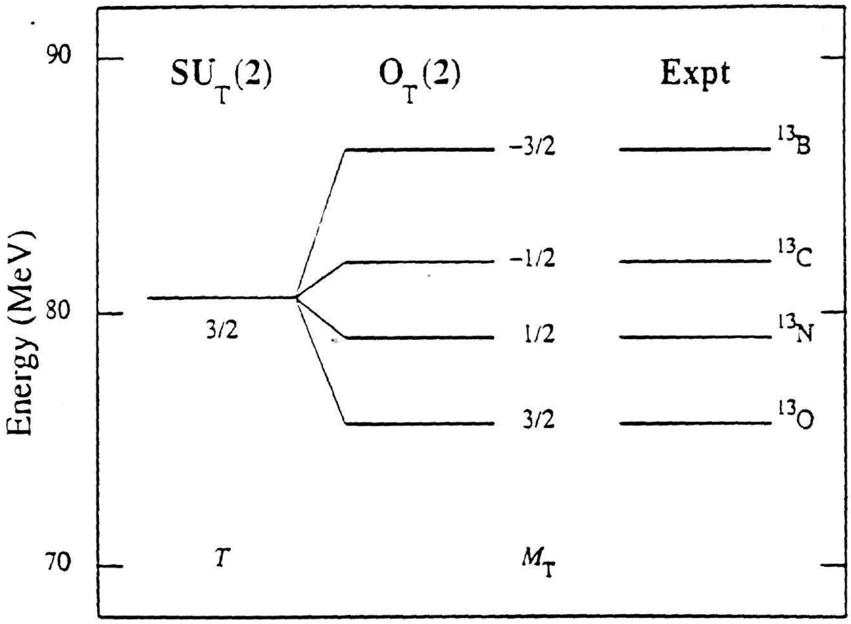


Figure 1: Binding energies of the $T = 3/2$ isobaric analog states with $J^\pi = 1/2^-$ in ^{13}B, ^{13}C, ^{13}N and ^{13}O. The column on the left is obtained for an exact $SU_T(2)$ *symmetry*, which predicts states with different M_T to be degenerate. The middle column is obtained in the case of an $SU_T(2)$ *dynamical symmetry*, equation (16) with $a = 80.59$, $b = -2.96$ and $c = -0.26$, in units of MeV.

while maintaining good quantum numbers Y, T and M_T, that is, it can be broken dynamically. Allowing only up to quadratic terms, a mass operator is found of the form

$$\begin{aligned} \hat{M} &= a + bC_1[U_Y(1)] + cC_2[U_Y(1)] \\ &\quad + dC_2[SU_T(2)] + eC_1[O_T(2)] + fC_2[O_T(2)] \end{aligned} \tag{18}$$

with the eigenvalues

$$M(Y,T,M_T) = a + bY + cY^2 + dT(T+1) + eM_T + fM_T^2 \tag{19}$$

Due to the electromagnetic interaction, $\hat{M}$ is not scalar in isospin, but contains also isospin vector and tensor terms (two last terms in (18)), for the reasons I have referred to in the previous example. Similarly, one assumes

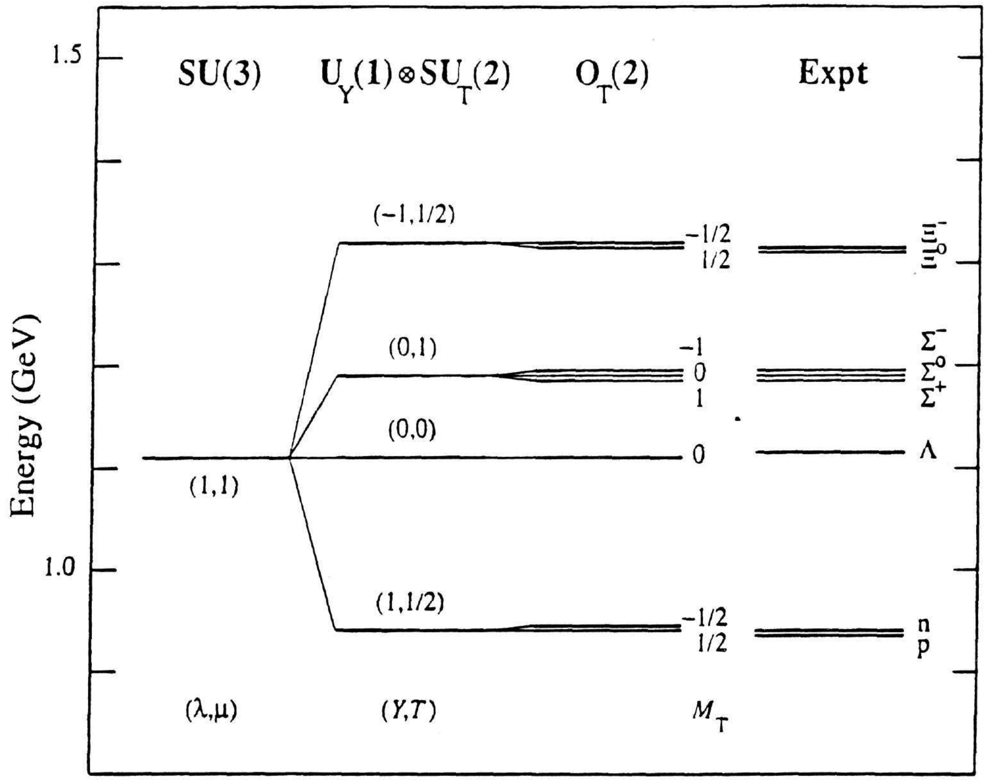


Figure 2: The mass spectrum of the SU(3) octuplet $(\lambda,\mu) = (1,1)$. The column on the left is obtained for an exact SU(3) *symmetry*, which predicts all masses to be the same, while the next two columns represent successive breakings of this symmetry in a dynamical manner. The column under $O_T(2)$ is obtained with equation (20) with $a = 1111.3$, $b = -189.6$, $c = -39.9$, $d = -3.8$ and $f = 0.9$, in units of MeV.

the strong interaction to have a certain tensor character under SU(3) and this leads to a relation between the coefficients c and d in (19), resulting in the SU(3) mass-splitting formula [13,14]

$$M'(Y,T,M_T) = a + bY + d\left(T(T+1) - \tfrac{1}{4}Y^2\right) + eM_T + fM_T^2 \qquad (20)$$

The process of successive symmetry breakings is illustrated in figure 2 with the example of the SU(3) octuplet $(\lambda,\mu) = (1,1)$, containing the neutron, the proton and the Λ, Σ and Ξ baryons.

We are now in a position to discuss one more concept frequently used in algebraic models, namely the one of a *dynamical algebra*, a single representation of which contains all states of the physical system under consideration. The idea is perhaps best understood with the help of the examples given above. For isobaric analog states in nuclei the dynamical algebra is $SU_T(2)$, since in that case we establish a relation between nuclear states contained in a single representation of $SU_T(2)$. The dynamical algebra in the particle physics example is SU(3), since all the particles which are simultaneously described belong to one (λ, μ) representation ((1,1) for the octuplet, (3,0) for the decuplet, etc.). A persistent theme in physics has been the search for larger dynamical algebras resulting in a more unified description of physical phenomena. This trend can be illustrated with figure 2. The near-equality of the masses of the neutron and the proton suggested the existence of isospin multiplets, later confirmed at higher energies for other 'elementary' particles. To establish, in turn, a relation between several of those multiplets, a larger dynamical algebra was needed, SU(3), which contains the isospin algebra $SU_T(2)$ as a subalgebra. This unification process did not stop with SU(3): to connect the different observed (λ, μ) multiplets, still larger dynamical algebras have been proposed (SU(4),...). However, typically the symmetry associated with a larger dynamical algebra will be more strongly broken.

In the next two sections I shall discuss the application of these ideas to the interacting boson–fermion model. There also one observes the gradual introduction of larger dynamical algebras and the trend towards a simultaneous description of larger systems.

4. Dynamical symmetries in odd-mass nuclei

Let me first recall how to characterize the interacting boson model for even–even nuclei from the perspective of symmetries. Unitary transformations among the six components occurring in the model ($s^\dagger$ and $d^\dagger_\mu, \mu = 0, \pm 1, \pm 2$) generate U(6). This algebra thus plays the role of dynamical algebra since all low-lying collective states of an even–even nucleus are described within a single (symmetric) representation $[N]$ of U(6), where N is the number of bosons. Imposing a U(6) symmetry for the description of such states would constitute a poor approximation since they would be predicted degenerate in energy. However, analogous to the examples given in section 3, the U(6) symmetry can be broken in a dynamical manner. It has been shown by Arima and Iachello [15,16,17] that three different types of dynamical symmetries occur in the interacting bosom model, associated with the three reduction schemes

$$\mathrm{U}(6) \supset \left\{ \begin{array}{c} \mathrm{U}(5) \supset \mathrm{O}(5) \\ \mathrm{SU}(3) \\ \mathrm{O}(6) \supset \mathrm{O}(5) \end{array} \right\} \supset \mathrm{O}(3) \tag{21}$$

where O(3) is the angular momentum algebra. Thus a situation arises where the original U(6) symmetry can be broken dynamically in three different ways each corresponding to a certain type of spectrum-generating hamiltonian. The process of symmetry breaking is analogous to that in the examples of section 3 and is illustrated in figure 3 with the O(6) limit of the interacting boson model: the various Casimir operators associated with O(6), O(5) and O(3) lift the original U(6) degeneracy to produce an energy spectrum close to that of ^{196}Pt. The breaking of symmetry occurs without admixing the wave functions and hence the labels associated with the different algebras in the third chain remain good quantum numbers. U(6) (as well as O(6) and O(5)) ceases to be a symmetry; only O(3) remains a true symmetry of the hamiltonian, reflecting its rotational invariance.

If one omits technical details, the algebraic treatment of odd-mass nuclei in the context of the interacting boson–fermion model can be explained in a completely analogous fashion. First we identify the dynamical algebra of the model: for the bosons this algebra remains U(6), commonly denoted as $\mathrm{U^B}(6)$ to stress its boson character. The fermion algebra is $\mathrm{U^F}(\Omega)$ where Ω is the degeneracy of the orbits available to the fermion (e.g., $\Omega = 4$ for a $j = 3/2$ orbit or $\Omega = 12$ for $j = 1/2, 3/2, 5/2$). Since boson and fermion generators commute, the total dynamical algebra of an odd-mass nucleus is $\mathrm{U^B}(6) \otimes \mathrm{U^F}(\Omega)$. Next degeneracy breaking terms are introduced by considering Casimir operators of subalgebras of the dynamical algebra. An example is shown in figure 4 in the case of $\mathrm{U^B}(6) \otimes \mathrm{U^F}(12)$, which applies to odd-neutron Pt isotopes. As in the application to even–even nuclei, also here a gradual breaking of the original symmetry produces an energy spectrum close to that of ^{195}Pt. Note the complexity of the final spectrum (as compared to previous examples) and the fact that a one-to-one correspondence between theory and experiment can be established. This correspondence is further confirmed by a study of E2 transition properties and nucleon transfer data [5].

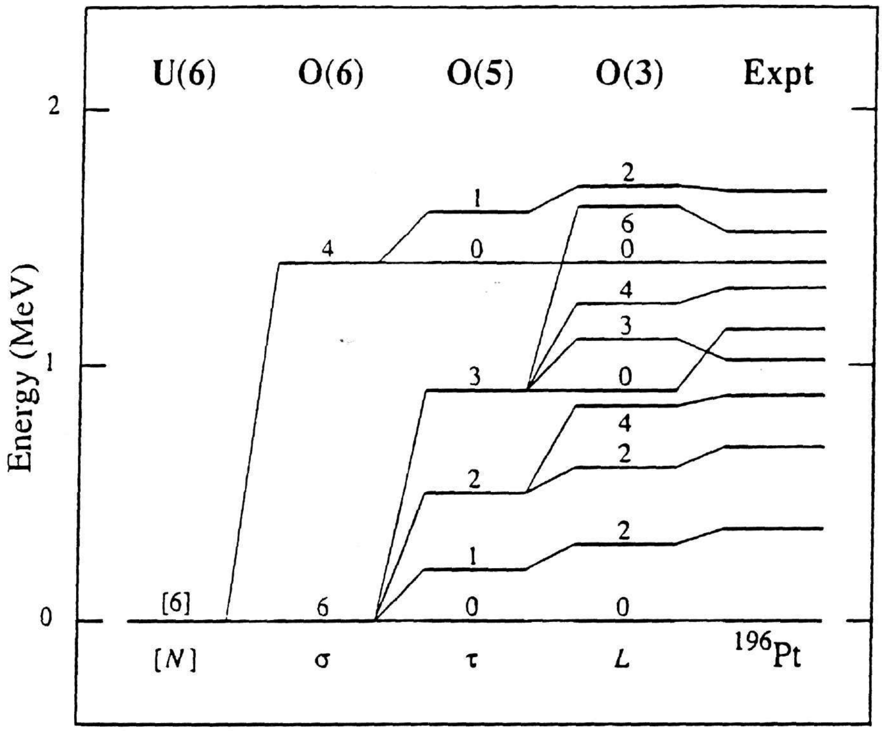


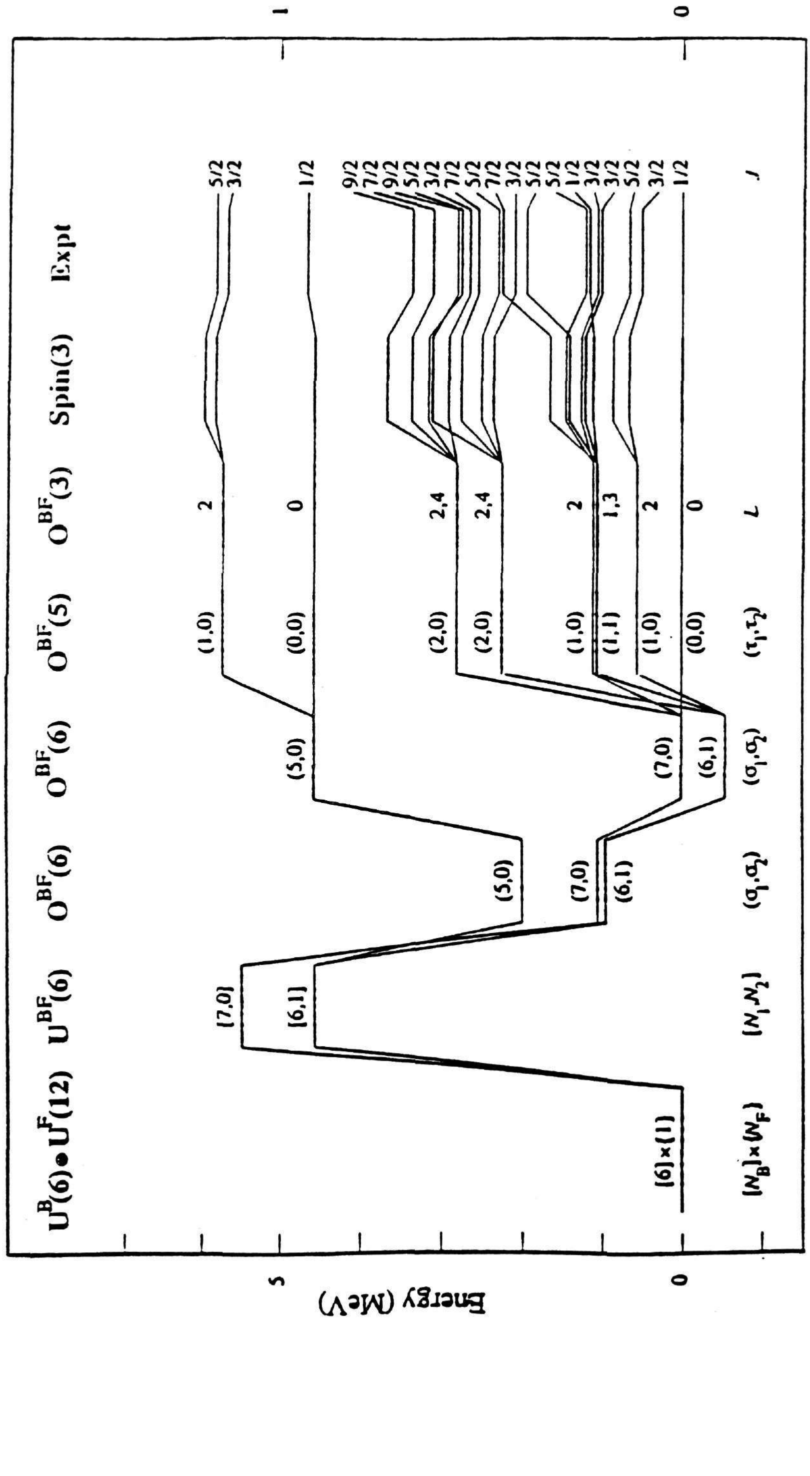
Figure 3: The observed energy spectrum of ^{196}Pt compared with a symmetry calculation in the O(6) limit of the interacting boson model. The original U(6) symmetry (left-hand side) of the hamiltonian is broken dynamically by including Casimir operators of O(6), O(5) and O(3). The labels associated with the various algebras are shown at the bottom.

5. Supersymmetries

We saw in the previous section how even–even nuclei in the interacting boson model and odd-mass nuclei in the interacting boson–fermion model can be treated in a unified framework using simple symmetry ideas. Schematically, states in such nuclei are described in terms of the following set of generators:

$$\begin{pmatrix} b^\dagger b & \\ & a^\dagger a \end{pmatrix} \tag{22}$$

where subscripts are omitted for simplicity. Even–even nuclei are described


U^B(6)•U^F(12)
U^BF(6)
O^BF(6)
O^BF(6)
O^BF(5)
O^BF(3)
Spin(3)
Expt
[N_B]×[N_F]
[N_1,N_2]
(σ_1,σ_2)
(σ_1,σ_2)
(τ_1,τ_2)
L
J
[6]×[1]
[7,0]
[6,1]
(5,0)
(7,0)
(6,1)
(1,0)
(0,0)
(2,0)
(1,1)
2
0
2,4
1,3
5/2
3/2
1/2
9/2
7/2
Energy (MeV)
5
0
1

in terms of the operators in the upper left-hand corner of (22) while odd-mass nuclei require both sets of generators. It is clear, however, that the operators (22) provide a *separate* description of even–even and odd-mass nuclei: although the treatment is similar in both cases, none of the operators (22) *connects* even–even and odd-mass states.

In 1981 Iachello [18] proposed an extension of the algebraic structure (22) by considering in addition operators that transform a boson into a fermion or *vice versa*:

$$\begin{pmatrix} b^\dagger b & b^\dagger a \\ a^\dagger b & a^\dagger a \end{pmatrix} \tag{23}$$

This set does not any longer form a classical Lie algebra which is defined entirely in terms of commutation relations. Instead, to define a closed algebraic structure, one needs to introduce an internal operation that corresponds to a mixture of commutation and anticommutation and the resulting algebra has been named a *graded* or *superalgebra*, denoted by $\mathrm{U}(n/m)$ where n and m are the dimensions of the boson and fermion parts.

To understand the purpose of introducing the supersymmetric generators $a^\dagger b$ or $b^\dagger a$ it is best to inspect their action on an even–even nucleus, say ^{194}Pt in the context of the $\mathrm{U^B}(6) \otimes \mathrm{U^F}(12)$ model discussed in section 4. We find

$$a^\dagger b\ {}^{194}_{78}\mathrm{Pt}_{116} \longrightarrow a^\dagger\ {}^{196}_{78}\mathrm{Pt}_{118} \longrightarrow {}^{195}_{78}\mathrm{Pt}_{117} \tag{24}$$

where bosons and fermions are assumed to have a neutron–hole character. Thus we find that the supersymmetric generators induce a connection between even–even and odd-mass nuclei; a description with $\mathrm{U}(6/\Omega)$ as a dynamical (super)algebra will automatically lead to a *simultaneous* treatment of such pairs of nuclei. This idea is illustrated schematically in figure 5 for the case of the Pt isotopes in U(6/12). A U(6/12) symmetry would predict

Figure 4: The observed spectrum of negative-parity states in ^{195}Pt compared with a symmetry calculation in the O(6) limit of $\mathrm{U^B}(6) \otimes \mathrm{U^F}(12)$. The original $\mathrm{U^B}(6) \otimes \mathrm{U^F}(12)$ symmetry (left-hand side) is broken dynamically by including Casimir operators of its subalgebras. The labels associated with the various algebras are shown at the bottom; the Spin(3) label J is shown under 'Expt'. The left-hand scale applies to the three left-most columns and the right-hand scale to all others. The spectrum under $\mathrm{O^{BF}}(6)$ is expanded and rescaled in the fourth column.

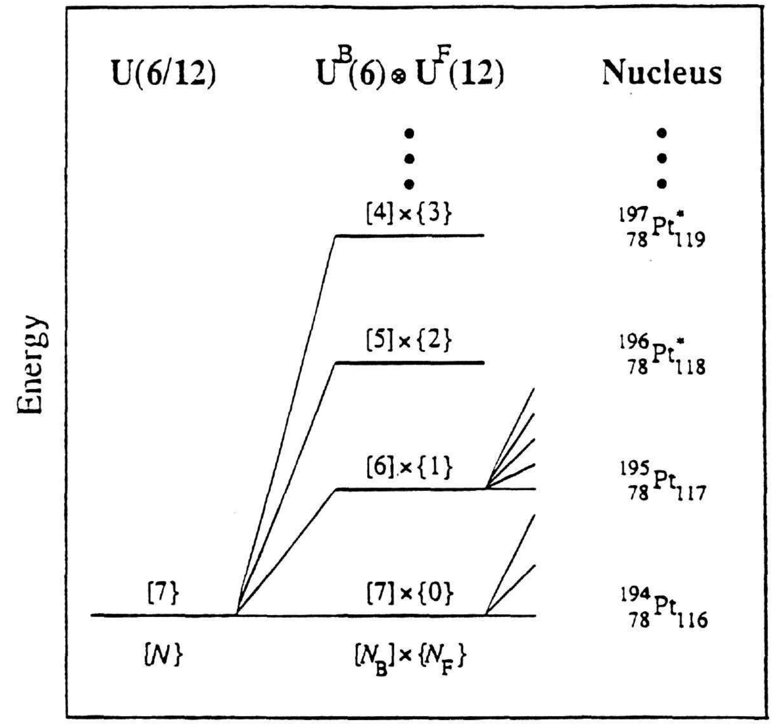


Figure 5: Schematic illustration of part of a U(6/12) supermultiplet in the Pt region. The supermultiplet is characterized by a supersymmetric representation with $\mathcal{N} = N_B + N_F = 7$. A breaking of the U(6/12) symmetry in a dynamical manner leads to a splitting in the binding energies of the different nuclei.

all states belonging to the supermultiplet degenerate in energy; this degeneracy is first lifted by including $U^B(6) \otimes U^F(12)$ invariants which correspond to nuclear binding energy terms. The analysis then proceeds as in section 4 with the inclusion of Casimir invariants of the lower algebras, as schematically indicated in figure 5. Thus supersymmetry introduces two additional ideas as compared to those already developed in the previous section: (i) It enables the analysis of binding energies. In view of the large number of parameters entering the model, a phenomenological study, however, is often difficult. (ii) It establishes a connection between the even–even and odd-mass nuclei belonging to the same supermultiplet. In practical terms it means that their spectra are calculated with the *same* hamiltonian, their electromagnetic transition properties with the *same* operator, etc. Several examples of supermultiplets of nuclei have been found; an overview is given in [5].

6. Conclusion

In this talk I have examined the interacting boson–fermion model from a specific viewpoint, namely its algebraic structure. The main message that I have tried to convey here is that a group-theoretical description of nuclei facilitates their unified treatment. This conclusion not only applies to nuclei but to many other physical systems as I have illustrated with some simple examples. In the interacting boson model the most striking example of this approach is the idea of supersymmetry: introducing a dynamical superalgebra naturally leads to a simultaneous description of even–even and odd-mass nuclei.

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