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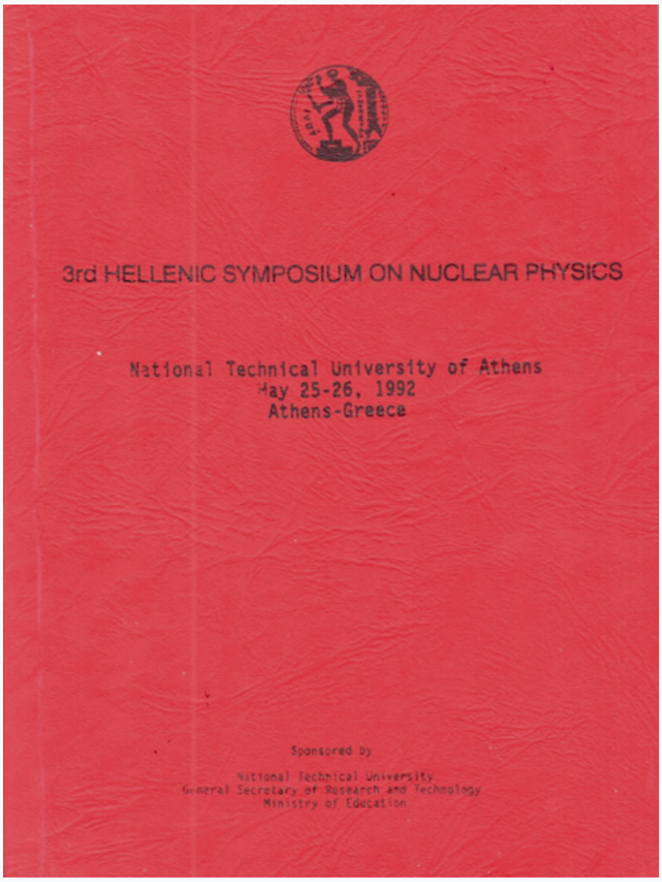


**Quantum algebraic symmetries in nuclear physics**

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# QUANTUM ALGEBRAIC SYMMETRIES IN NUCLEAR PHYSICS *

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**Abstract**

The pairing correlations in a single-j nuclear shell are considered. It is proven that a simple boson mapping in terms of q-deformed bosons exists, which reproduces correctly both the commutation relations and the energy up to first order corrections, the parameter $q$ being connected to the size of the shell. An exact solution in terms of a generalized deformed oscillator is also found.

## 1. Introduction

Quantum algebras (also called quantum groups), which from the mathematical point of view are q-deformations of the universal enveloping algebras of the corresponding Lie algebras, are recently receiving much attention in physics, especially after the introduction of the q-deformed harmonic oscillator [1, 2]. Initially used for solving the quantum Yang Baxter equation, quantum algebras are now finding applications in conformal field theory, quantum gravity, quantum optics, as well as in the study of spin chains. Their first applications in rotational spectra of deformed nuclei [3, 4], superdeformed nuclei [5] and diatomic molecules [6], as well as in vibrational spectra of diatomic molecules [7, 8], have been briefly reviewed during the 2nd Hellenic Symposium on Nuclear Physics last year [9].

* Presented by Dennis Bonatsos

Much progress has been made in these directions since then. Concerning deformed nuclei, not only the energy levels, but also the probabilities of the electric quadrupole transitions connecting them have been found to be described well by the $SU_q(2)$ symmetry [10], while for vibrational spectra of diatomic molecules an accurate description can be obtained through the use of a suitable generalized deformed oscillator [11–13]. The construction of classical potentials giving the same spectrum as the q-deformed harmonic oscillator [14] and the various versions of q-deformed anharmonic oscillators used in molecular physics [15–17] has also been advanced.

The q-deformed bosons used in the above mentioned applications satisfy commutation relations which differ from the standard boson commutation relations, in which they reduce in the limit in which the deformation parameter $q$ goes to 1. On the other hand, it is well-known in nuclear physics that correlated fermion pairs in a single-j shell [18] or several non-degenerate j-shells [19–21] satisfy commutation relations which resemble boson commutation relations including corrections due to the presence of the Pauli principle. This fact has been the cause for the development of boson mapping techniques (for a recent review see [21] and references therein), by which the description of systems of fermions in terms of bosons is achieved. In recent years boson mappings have attracted additional attention in nuclear physics as a necessary tool in providing a theoretical justification for the success of the phenomenological Interacting Boson Model and its various extentions (see [22, 23] for recent overviews), in which low lying collective states of medium and heavy mass nuclei are described in terms of bosons.

From the above observations it is clear that both q-bosons and correlated fermion pairs satisfy commutation relations which resemble the standard boson commutation relations but they deviate from them, due to the q-deformation in the former case and to the Pauli principle in the latter. A question is thus created: Are q-bosons suitable for the approximate description of correlated fermion pairs? In particular, is it possible to construct a boson mapping in which correlated fermion pairs are mapped onto q-bosons, in a way that the q-boson operators approximately satisfy the same commutation relations as the correlated fermion pair operators? In this talk we show for the simple case of SU(2) that such a mapping is possible. In section 2 we shall give an approximate solution to the problem, while in section 3 an exact solution will be constructed. Finally in section 4 discussion of the present results and plans for future work will be given.

**2. Approximate description of the pairing correlations in a single-j shell in terms of q-deformed bosons**

Let us consider the single-j shell model [18–21]. One can define fermion pair and multipole operators as

$$A^{+}_{JM} = \frac{1}{\sqrt{2}} \sum_{mm'} (jmjm'|JM) a^{+}_{jm} a^{+}_{jm'}, \tag{1}$$

$$B_{JM} = \frac{1}{\sqrt{2J+1}} \sum_{mm'} (jmj-m'|JM)(-1)^{j-m'} a^{+}_{jm} a_{jm'}, \tag{2}$$

with the following definitions

$$A_{JM} = [A^{+}_{JM}]^{+}, \quad B^{+}_{JM} = [B_{JM}]^{+}. \tag{3}$$

In the above $a^{+}_{jm}$ ($a_{jm}$) are fermion creation (annihilation) operators and $(jmjm'|JM)$ are the usual Clebsch–Gordan coefficients.

The pair and multipole operators given above satisfy the commutation relations of the SO(2(2j+1)) algebra. In the present talk we will restrict ourselves to fermion pairs coupled to angular momentum zero. The relevant commutation relations take the form

$$[A_0, A^{+}_0] = 1 - \frac{N_F}{\Omega}, \qquad [\frac{N_F}{2}, A^{+}_0] = A^{+}_0, \qquad [\frac{N_F}{2}, A_0] = -A_0, \tag{4}$$

where $N_F$ is the number of fermions, $2\Omega = 2j+1$ is the size of the shell, and

$$B_0 = N_F/\sqrt{2\Omega}. \tag{5}$$

With the identifications

$$J_{+} = \sqrt{\Omega} A^{+}_0, \qquad J_{-} = \sqrt{\Omega} A_0, \qquad J_0 = \frac{N_F - \Omega}{2}, \tag{6}$$

eq. (4) takes the form of the usual SU(2) commutation relations

$$[J_{+}, J_{-}] = 2J_0, \quad [J_0, J_{+}] = J_{+}, \quad [J_0, J_{-}] = -J_{-}. \tag{7}$$

The simplest pairing Hamiltonian one can consider has the form ([24] and references therein)

$$H = -G\Omega A_0^+ A_0. \tag{8}$$

The Casimir operator of SU(2) can be written as

$$\{A_0^+, A_0\} + \frac{\Omega}{2}(1 - \frac{N_F}{\Omega})^2 = \frac{\Omega}{2} + 1, \tag{9}$$

while the pairing energy takes the form

$$\frac{E}{(-G\Omega)} = \frac{N_F}{2} - \frac{N_F^2}{4\Omega} + \frac{N_F}{2\Omega}. \tag{10}$$

Our aim is to check if there is a boson mapping for the operators $A_0^+$, $A_0$ and $N_F$ in terms of q-deformed bosons, having the following properties:

i) The mapping is simpler than the usual Holstein–Primakoff mapping [21], i.e. to each fermion pair operator $A_0^+$, $A_0$ corresponds a bare q-boson operator and not a boson operator accompanied by a square root (the Pauli reduction factor).

ii) The commutation relations (4) are satisfied up to a certain order.

ii) The pairing energies of eq. (10) are reproduced up to the same order.

In recent work q-numbers are defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{11}$$

where $q$ can be real ($q = e^\tau$, where $\tau$ real) or a phase ($q = e^{i\tau}$, with $\tau$ real). The q-deformed harmonic oscillator [1, 2] is defined in terms of the creation and annihilation operators $a^+$ and $a$ and the number operator $N$, which satisfy the commutation relations

$$[N, a^+] = a^+, \quad [N, a] = -a, \quad aa^+ - q^{\mp 1} a^+ a = q^{\pm N}. \tag{12}$$

An immediate consequence of (12) is that

$$a^+ a = [N], \quad aa^+ = [N+1]. \tag{13}$$

The Hamiltonian of the q-deformed harmonic oscillator is

$$H = \frac{\hbar\omega}{2}(aa^+ + a^+ a), \tag{14}$$

and its eigenvalues are

$$E(n) = \frac{\hbar\omega}{2}([n] + [n+1]). \tag{15}$$

For $q$ being a phase, the commutator of $a$ and $a^+$ takes the form

$$[a, a^+] = [N+1] - [N] = \frac{\cos\frac{(2N+1)\tau}{2}}{\cos\frac{\tau}{2}}. \tag{16}$$

In physical situations $\tau$ is expected to be small (i.e. of the order of 0.01), as in the cases of refs. [3–9]. Therefore in eq. (16) one can take Taylor expansions of the functions appearing there and thus find an expansion of the form

$$[a, a^+] = 1 - \frac{\tau^2}{2}(N^2 + N) + \frac{\tau^4}{24}(N^4 + 2N^3 - N) - \ldots . \tag{17}$$

We remark that the first order corrections contain not only a term proportional to $N$, but in addition a term proportional to $N^2$, which is larger than $N$. Thus one cannot make the simple mapping

$$A_0 \to a, \quad A_0^+ \to a^+, \quad N_F \to 2N, \tag{18}$$

because then one cannot get the commutation relation (4) correctly up to the first order of the corrections. The same problem appears in the case that $q$ is real as well. In addition, by making the simple mapping of eq. (18) the pairing Hamiltonian can be written as

$$\frac{H}{-G\Omega} = a^+ a = [N]. \tag{19}$$

In the case of small $\tau$, one can again take Taylor expansions of the trigonometric (hyperbolic) functions appearing in the definition of the q-numbers for $q$ being a phase (real) and thus obtain the following expansion

$$[N] = N \pm \frac{\tau^2}{6}(N - N^3) + \frac{\tau^4}{360}(7N - 10N^3 + 3N^5) \pm \frac{\tau^6}{15120}(31N - 49N^3 + 21N^5 - 3N^7) + \ldots, \tag{20}$$

where the upper (lower) sign corresponds to $q$ being a phase (real). We remark that while the first order corrections in eq. (10) are proportional to $N_F^2$ and $N_F$, here the first order corrections are proportional to $N$ and $N^3$. Thus neither the pairing energies can be reproduced correctly by this mapping.

However, a different version of the q-harmonic oscillator can be obtained by defining (see [24] and references therein) the operators $b$, $b^+$ through the equations

$$a = q^{1/2} b q^{-N/2}, \quad a^+ = q^{1/2} q^{-N/2} b^+. \tag{21}$$

Eq. (12) then gives

$$[N, b^+] = b^+, \quad [N, b] = -b, \quad bb^+ - q^2 b^+ b = 1. \tag{22}$$

By using the symbol $Q = q^2$ and introducing the Q-number

$$[x]_Q = \frac{Q^x - 1}{Q - 1}, \tag{23}$$

we find the analog of the eq. (13)

$$b^+ b = [N]_Q, \quad bb^+ = [N+1]_Q. \tag{24}$$

The Hamiltonian of the corresponding deformed harmonic oscillator has the form

$$H = \frac{\hbar\omega}{2}(bb^+ + b^+ b), \tag{25}$$

the eigenvalues of which are

$$E(n) = \frac{\hbar\omega}{2}([n]_Q + [n+1]_Q). \tag{26}$$

It should be noticed at this point that the definition of q-number given in eq. (23) was historically the first to be introduced in the framework of q-analysis. For convenience from now on we will call the numbers of eq. (23) the Q-numbers, while the numbers of eq. (11) we will call the q-numbers.

From the above relations, it is clear that the following commutation relation holds

$$[b, b^+] = [N+1]_Q - [N]_Q = Q^N. \tag{27}$$

Defining $Q = e^T$ this can be written as

$$[b, b^+] = 1 + TN + \frac{T^2 N^2}{2} + \frac{T^3 N^3}{6} + \ldots. \tag{28}$$

We remark that the first order correction is proportional to $N$. Thus, by making the boson mapping

$$A_0^+ \to b^+, \quad A_0 \to b, \quad N_F \to 2N, \tag{29}$$

one can satisfy eq. (4) up to the first order of the corrections by determining $T = -2/\Omega$.

We should now check if the pairing energies (eq. (10)) can be found correctly up to the same order of approximation when this mapping is employed. The pairing Hamiltonian in this case takes the form

$$\frac{H}{-G\Omega} = b^+ b = [N]_Q. \tag{30}$$

Defining $Q = e^T$ it is instructive to construct the expansion of the Q-number of eq. (23) in powers of $T$. Assuming that $T$ is small and taking Taylor expansions in eq. (23) one finally has

$$[N]_Q = N + \frac{T}{2}(N^2 - N) + \frac{T^2}{12}(2N^3 - 3N^2 + 1) + \frac{T^3}{24}(N^4 - 2N^3 + N^2) + \ldots \tag{31}$$

Using the value of the deformation parameter $T = -2/\Omega$, determined above from the requirement that the commutation relations are satisfied up to first order corrections, the pairing energies become

$$\frac{E}{-G\Omega} = N - \frac{N^2 - N}{\Omega} + \frac{2N^3 - 3N^2 + 1}{3\Omega^2} - \frac{N^4 - 2N^3 + N^2}{3\Omega^3} + \ldots. \tag{32}$$

The first two terms in the rhs of eq. (32), which correspond to the leading term plus the first order corrections, are exactly equal to the pairing energies of eq. (10), since $N_F \to 2N$. We therefore conclude that through the boson mapping of eq. (29) one can both satisfy the fermion pair commutation relations of eq. (4) and reproduce the pairing energies of eq. (10) up to the first order corrections.

### 3. Exact description of the pairing correlations in a single-j shell in terms of a generalized deformed oscillator

In the previous section we found that an approximate description of the pairing correlations in a single-j shell can be given in terms of Q-deformed bosons. In this section an exact description will be given, using a generalized deformed oscillator.

Instead of using deformed bosons satisfying eq. (12) or eq. (22), one can use bosons satisfying the commutation relation [11, 25]

$$f(aa^+) - f(a^+a) = 1, \tag{33}$$

where $f(x)$ is a real analytic function defined on the real positive axis. In the case of the usual harmonic oscillator one has $f(x) = x$, which leads to the usual boson commutation relation $[a, a^+] = 1$.

The idea of the generalized deformed oscillator has been described in detail in [11, 12]. The energy operator is

$$H = \frac{A}{2}(aa^+ + a^+a), \tag{34}$$

while its eigenvalues are

$$E_n = \frac{A}{2}(F(n+1) + F(n)), \tag{35}$$

where $F = f^{-1}$ is the structure function determining the properties of the oscillator.

We now apply this procedure in the case of the pairing in a single-j shell mentioned before. The boson number is half the fermion number, i.e. $N = N_F/2$. Then eq. (10) can be written as

$$\frac{E}{-G\Omega} = N - \frac{N^2}{\Omega} + \frac{N}{\Omega}. \tag{36}$$

Taking into account refs [11, 12] and eq. (36) one has

$$a^+a = F(N) = N - \frac{N^2}{\Omega} + \frac{N}{\Omega}. \tag{37}$$

In addition one has

$$aa^+ = F(N+1) = N + 1 - \frac{(N+1)^2}{\Omega} + \frac{N+1}{\Omega}. \tag{38}$$

What we have constructed is a boson mapping for the operators $A_0$, $A_0^+$, $N_F$:

$$A_0 \to a, \quad A_0^+ \to a^+, \quad N_F \to 2N. \tag{39}$$

From eq. (37) it is clear that this mapping gives the correct pairing energy. In addition one has

$$[a, a^+] = F(N+1) - F(N) = 1 - \frac{2N}{\Omega}, \tag{40}$$

in agreement to eq. (4). Thus the correct commutation relations are also obeyed.

An exact hermitian boson mapping for the SU(2) algebra is known to have the form [18–21]

$$A_0^+ = a_0^+\sqrt{1-\frac{n_0}{\Omega}}, \quad A_0 = \sqrt{1-\frac{n_0}{\Omega}}a_0, \quad N_F = 2n_0, \tag{41}$$

where $a_0^+$ ($a_0$) are boson creation (annihilation) operators carrying angular momentum zero and $n_0$ is the number of these bosons. In this mapping the Pauli principle effects are carried by the square roots accompanying the ordinary boson operators, while in the mapping of eq. (39) the Pauli principle effects are "built in" the deformed bosons.

The generalized oscillator obtained here has energy spectrum

$$E_N = \frac{1}{2}(F(N)+F(N+1)) = N+\frac{1}{2}-\frac{N^2}{\Omega}, \tag{42}$$

which is the spectrum of an anharmonic oscillator.

It is worth mentioning at this point that the energy spectrum of the generalized oscillator corresponding to the pairing correlations (eq. (42)) can be rewritten as

$$E_N = \frac{2}{\Omega}\left(-\frac{1}{8}+\frac{\Omega+1}{2}(N+\frac{1}{2})-\frac{1}{2}(N+\frac{1}{2})^2\right). \tag{43}$$

On the other hand, it is known that for the modified Pöschl–Teller potential

$$V(x) = D\tanh^2(x/R), \tag{44}$$

the energy spectrum is given by [12]

$$E_N = \frac{\hbar^2}{mR^2}\left(-\frac{1}{8}+\frac{1}{2}\sqrt{8mDR^2/\hbar^2+1}\;\;(N+\frac{1}{2})-\frac{1}{2}(N+\frac{1}{2})^2\right). \tag{45}$$

It is thus clear that the energy spectrum of the generalized oscillator studied here can be obtained from the modified Pöschl–Teller potential for special values of the potential depth $D$.

In conclusion, we have constructed a generalized deformed oscillator which satisfies the same commutation relations as fermion pair and multipole operators of zero angular momentum in a single-j shell, and, in addition, reproduces the pairing energy of this shell exactly. We have thus demonstrated that an exact hermitian boson mapping of a system of angular-momentum-zero fermion pairs in terms of bare deformed bosons can be constructed, while in the usual case the ordinary bosons are accompanied by square roots due to the Pauli principle effects. The oscillator corresponding to the pairing problem has a spectrum which can be reproduced up to first order perturbation theory by a harmonic oscillator with an $x^4$ anharmonicity [16]. The construction of a generalized deformed oscillator corresponding to any anharmonic oscillator has also been achieved [16].

## 4. Discussion

The results obtained in this letter indicate that deformed bosons might be a convenient tool for describing systems of fermion pairs under certain conditions. The generalization of the results obtained here to fermion pairs of nonzero angular momentum, which will allow for a fuller treatment of the single-j shell in terms of deformed bosons, is under investigation. Another interesting direction is the construction of exactly soluble models having quantum algebraic symmetries. The construction of a q-generalization of the full IBM [22–23] has not been carried out yet. However, a two-dimensional toy model, bearing the main characteristics of IBM and having an $SU_q(3)$ symmetry, has been studied in detail [26].

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