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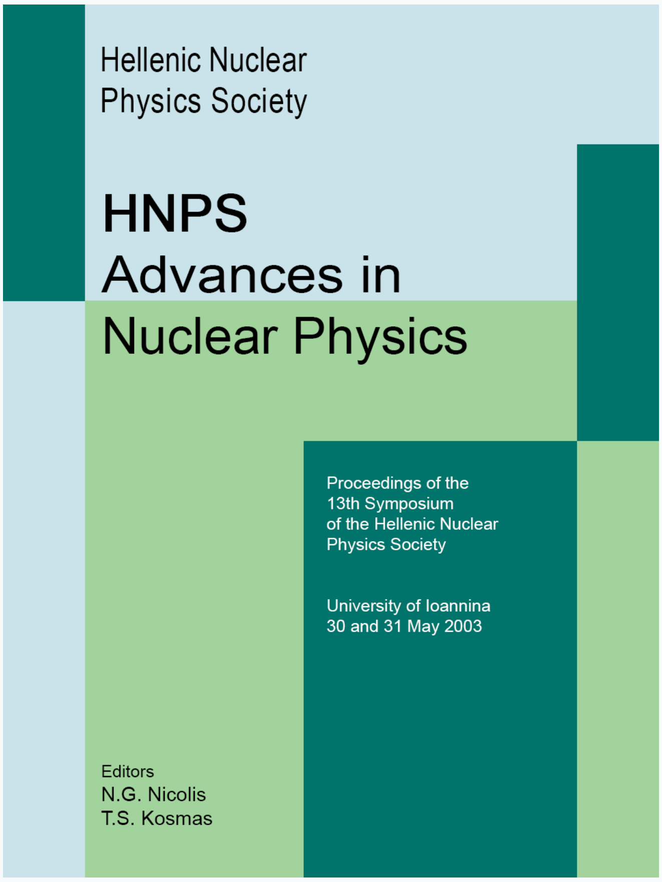

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A Study of τ^- and μ^- in the field of nuclei using Neural Network techniques

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Abstract

The rate of a heavy lepton (muon or tau) capture by nuclei as well as the heavy lepton to electron conversion rate can be calculated when the heavy lepton wavefunction is known. Analytical calculation of the wavefunction of any of these leptons around any nucleus is not feasible owning to their small Bohr radii, on the one hand, and to the finite nuclear extend on the other. A new numerical calculation algorithm is proposed hereby, which makes use of the concept of neural networks. The main advantage of this new technique is that the wave function is produced analytically as a sum of sigmoid functions.

1 Introduction

1. Introduction

Muon, a lepton with charge -e and mass 207 times that of e^-, can be bound to atomic nuclei forming muonic states similar to the electronic ones. The Bohr radius of a muonic state is much smaller from that of the corresponding electronic state by the ratio m_μ/m_e. For the same reason the third lepton of the family, tau, has an orbit of Bohr radius 1777 times smaller from the corresponding of the electron. There is thus significant overlap between the bound state wavefunctions of any of these heavy- leptons and the nucleus. Thus, the calculation of the bound state wavefunction of a heavy-lepton requires the knowledge of the distribution of the nuclear charge.

The bound heavy-lepton wavefunction can be used in the evaluation of the rates of certain heavy-lepton to electron transitions, either predicted by the standard model:

$$\mu_b^- \rightarrow e^- + \overline{\nu}_e + \nu_\mu \tag{1}$$

or exotic:

$$\mu_b^- + (A, Z) \rightarrow e^- + (A, Z)^* \tag{2}$$

$$\mu_b^- + (A, Z) \rightarrow e^+ + (A, Z-2) \tag{3}$$

and it could also be used in the calculation of the rate of the muon capture:

$$\mu_b^- + (A, Z) \rightarrow (A, Z-1) + \nu_\mu \tag{4}$$

the exotic decays 2 and 3 haven't yet been detected as they violate certain leptonic numbers. There are however several experiments looking for them. Thus SINDRUM II [1] has announced upper limit for the branching ratio of 2 the value 5×10^{-13}, while TRIUMF has announced upper limit for the branching ratio of 3 the value 9×10^{-12}.

The calculation of the rates of the above processes, requires the wavefunctions of these bound leptons. Thus, in the rest of this work we will try to obtain these wavefunctions solving the quantum mechanical equations using Neural Network techniques.

2 Matrix element formulation of the problem

In a bound heavy lepton to electron transition, the nucleus, initially in the ground state $|i/rangle$ gets exited to the final state $|f\rangle$. The matrix element of this transition is:

$$M_\alpha^{(\tau)} = \langle f | sum_{j=1}^{A} \Theta_\alpha^\tau(j) e^{-i\hat{\mathbf{q}}\cdot\mathbf{r_j}} \Phi(\mathbf{r_j}) |\mathbf{i}\rangle \tag{5}$$

,

where $\Theta_\alpha^\tau(j)$ are spin and isospin dependent functions describing the relevant process, $\hat{\mathbf{q}}$ a unit vector in the direction of the nuclear momentum transfer, $\Phi(\mathbf{r_j})$ the initial heavy lepton state in the nuclear Coulomb potential. The matrix element $M_\alpha^{(\tau)}$ can be approximately calculated by factorizing it as follows:

$$\overline{M}_\alpha^{(\tau)} = \langle \Phi_\mu^{1s} \rangle \langle f | sum_{j=1}^{A} \Theta_\alpha^\tau(j) e^{-i\hat{\mathbf{q}}\cdot\mathbf{r_j}} | i \rangle = \langle \Phi_\mu^{1s} \rangle M_\alpha^{\prime(\tau)} \tag{6}$$

i.e. as a product of a suitably averaged wavefunction and a matrix element dependent purely on nuclear properties. Our method of producing the matrix element is by calculating the exact initial and final state wavefunction which is then used as input in 5.

3 Solution of the Schrödinger and Dirac equations using Neural Networks

We tried to solve both, the Schroedinger and Dirac equations, to obtain the radial parts $u(r) = rR(r)$ of the heavy lepton wavefunctions. However, since the Bohr radii of these leptons are much smaller than the electron's Bohr radius, the potential V(r) has the following features: i) the presence of the other electrons has a negligible influence to it and ii) it is strongly dependent on the nuclear charge distribution. Thus for the Schroedinger wave equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u(r) + V(r)u(r) = Eu(r) \tag{7}$$

under the boundary condition u(0)=0 and asymptotic behaviour $e^{-\beta r}$, the potential function has two contributions. The main one is due to the Coulomb field:

$$V_e(r) = -e\int_0^i nfty \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}. \tag{8}$$

where m is the reduced mass of the lepton-nucleus system. The other contribution is due to the vacuum polarization:

$$V_L(r) = -2\pi\frac{e^2}{r}\int_0^i nfty\rho(r')r'[|r-r'|(ln(C|r-r'|\lambda_e)-1)-(r+r')(ln(C(r+r')/\lambda_e-1))dr'eq3) \tag{9}$$

u(r) satisfies both the equation and the boundary conditions, when E has the ground state value:

$$N_0\frac{\hbar^2}{2m}\int_0^i nfty \left(\frac{du(r)^2}{dr} + V(r)u(r)^2\right) dr \tag{10}$$

where N_0, the normalization constant, has the value:

$$\left(\int_0^i nftyu(r)^2dr\right)^{-1} \tag{11}$$

Dirac wave equation for the radial parts u_1 and u_2 of the Dirac spinor ψ is written as a system of two equations:

$$\frac{d}{dr}u_1 = -\frac{\kappa}{r}u_1 + (W - V + m)u_2 \tag{12}$$

$$\frac{d}{dr}u_2 = -(W - V - m)u_1 + \frac{\kappa}{r}u_2 \tag{13}$$

where W is the energy, V(r) the potential function and m the reduced mass. The boundary conditions for the Dirac eq. are the same as for the Schroedinger eq.

Since nucleus has to be regarded as an extended charge distribution in order to calculate the Coulomb potential using 8 above, a form of $\rho(r)$ is needed. We can fit the experimental data for $\rho(r)$ using either the harmonic oscillator model:

$$\rho_{p(n)}(r) = \rho_0\left(1 + \alpha\left(\frac{r}{a}\right)^2\right)exp\left(-\left(\frac{r}{a}\right)^2\right) \tag{14}$$

or the 2 parameter Fermi one:

$$\rho_{p(n)}(r) = \frac{\rho_0}{1 + exp\left(\left(r - c_{p(n)}\right)/z_{p(n)}\right)} \tag{15}$$

There are many numerical techniques to solve partial differential equations. The Neural Network technique[2] which we will make use of is a modern one. It uses a powerful minimization program, MERLIN[3], and produces an analytic solution to a boundary value problem at any desired precision. However, in its present state of development, it produces results for the ground state only.

One starts with a trial wavefunction having the form:

$$u_t(r) = re^{-\beta r} N(r, \mathbf{u}, \mathbf{v}, \mathbf{w}), \beta > \mathbf{0}, \tag{16}$$

where the factor $e^{-\beta r}$ is necessary to ensure that the wavefunction dies away exponentially and the factor N, which depends on r and on the parameters **u**, **v**, **w**, is the neural network, or perceptron, function.

Defining a sigmoid transfer function of the form:

$$\sigma(x) = \frac{1}{1 + e^{-x}} \tag{17}$$

we define the one hidden layer perceptron as:

$$N(r, \mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{v_i} \sigma(\mathbf{z_i}) \tag{18}$$

where v_i, the i component of $\mathbf{v}$, is the weight from the hidden unit i to the output and $z_i = \sum_{j=1}^{n} w_{ij} r_j + u_i$, where w_{ij} is the weight from hidden unit j to hidden unit i and u_i the bias of hidden unit i. r_j are the points of a subdivision or the radial variable (i.e. the input units of the perceptron). The subdivision starts at r=0 and ends at a point where u(r) almost vanishes.

As a first attempt in solving the Dirac or the Schroedinger equations, we tried a diagonal form for w_{ij}. Thus:

$$N(r, \mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{\mathbf{j=1}}^{\mathbf{m}} \mathbf{v_j} \sigma(\mathbf{w_j r} + \mathbf{u_j}) \tag{19}$$

Then we apply the collocation technique[4]. We search for values of the parameters **u**, **v**, **w** for which the error function

$$\frac{\sum (Hu_t(r_i) - \epsilon u_t(r_i))^2}{\int_0^i nftyu_t^2(r)dr} \tag{20}$$

assumes a minimum. The value of ϵ is computed as:

$$\epsilon = \frac{\int_0^i nftyu_t(r)Hu_t(r)dr}{\int_0^i nftyu_t^2(r)dr} \tag{21}$$

The closest the error function is to zero, the better the approximation. For making the minimization, we used the MERLIN MCL minimization algorithm[3].

4 Application of the Dirac eq. Solution in the calculation of the decay rates

The decay rates ω of the above processes make use of the following 5 overlap integrals[5]

$$D = \frac{4}{\sqrt{2}} m_\mu \int_0^i nftydrr^2 [-E(r)] (g_e^- f_\mu{}^- + f_e^- g_\mu{}^-) \tag{22}$$

$$S^{(p)} = \frac{1}{2\sqrt{2}} \int_0^i nftydrr^2 Z \rho^{(p)} (g_e^- g_\mu{}^- - f_e^- f_\mu{}^-) \tag{23}$$

$$S^{(n)} = \frac{1}{2\sqrt{2}} \int_0^i nftydrr^2 (A - Z) \rho^{(n)} (g_e^- g_\mu{}^- - f_e^- f_\mu{}^-) \tag{24}$$

$$V^{(p)} = \frac{1}{2\sqrt{2}} \int_0^i nftydrr^2 Z \rho^{(p)} (g_e^- g_\mu{}^- + f_e^- f_\mu{}^-) \tag{25}$$

$$V^{(n)} = \frac{1}{2\sqrt{2}} \int_0^i nftydrr^2 (A - Z) \rho^{(n)} (g_e^- g_\mu{}^- + f_e^- f_\mu{}^-) \tag{26}$$

where $g_\mu^-(r)\chi_{-1}^{\pm 1/2}(\theta\phi)$ and $if_\mu^-(r)\chi_1^{\pm 1/2}(\theta,\phi)$ are the components of the initial muon state and $g_e^\pm(r)\chi_{-1}^{\pm 1/2}(\theta,\phi)$ and $if_e^\pm(r)\chi_1^{\pm 1/2}(\theta,\phi)$ the components of the final electron state.

The diagram below shows the muon ground state wavefunctions for ^208Pb. The full lines correspond to the 2 components of the Dirac eq. solution. The dashed line corresponds to the Shroedinger eq. solution.

Substituting $f_{e(\mu)}^\pm$ and $g_{e(\mu)}^\pm$ in the overlap integrals above, we can produce their values for all of the elements of the Periodic Table. These results are represented in the diagram bellow[5].

5 Conclusions

The bound states of muon and tau leptons are significant in present state research. They take part in various semileptonic processes the study of which is useful in the research of the electroweak interactions.

Precise calculation of the heavy lepton - nucleus overlap integrals, which are needed for the calculation of the decay rates of these leptons, requires knowledge of their bound state wavefunctions. Although the heavy lepton wavefunctions can be calculated numerically, the use of neural network techniques offers some advantages.

The calculation can be improved if we take into account certain corrections (nuclear polarization, relativistic corrections etc.)

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