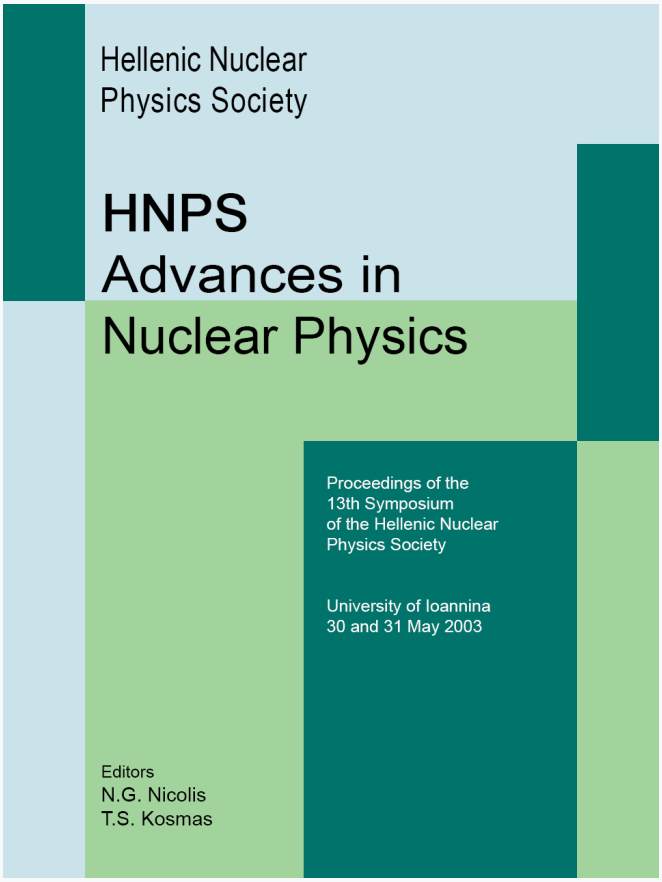
HNPS Advances in Nuclear Physics

Vol 12 (2003)

HNPS2003



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doi: 10.12681/hnps.3344

HNPS Advances in Nuclear Physics vol. 12, pp. 51-60 (2003)
HNPS2003
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Parameter-Independent Symmetries in Nuclear Structure

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Abstract

The E(5) symmetry describes nuclei related to the U(5)–SO(6) phase transition, while the X(5) symmetry is related to the U(5)–SU(3) phase transition. A chain of potentials interpolating between the U(5) symmetry of the 5-dimensional harmonic oscillator and the E(5) symmetry is considered. Parameter independent predictions for the spectra and B(E2) values of nuclei with $R_4 = E(4)/E(2)$ ratios 2.093, 2.135, and 2.157 (compared to the ratio 2.00 of the U(5) case and the ratio 2.20 of the E(5) case) are derived numerically and compared to existing experimental data, suggesting several new experiments. Furthermore, an exactly soluble model with $R_4 = 2.646$ is constructed and a chain of potentials interpolating between this new model and the X(5) symmetry is considered. Parameter independent predictions for the spectra and B(E2) values of nuclei with R_4 ratios 2.769, 2.824 and 2.852 (compared to the ratio 2.904 of the X(5) case) are derived numerically and compared to existing experimental data, suggesting several new experiments.

1 Introduction

The recently introduced E(5) [1] and X(5) [2] models have attracted considerable interest, since they are considered to be related to the critical points of the phase transitions from U(5) (vibrational) to O(6) (γ-unstable) nuclei and from U(5) to SU(3) (prolate deformed) nuclei respectively. Both models originate (under certain simplifying assumptions) from the Bohr collective Hamiltonian [3], which is known to possess the U(5) symmetry of the five-dimensional (5-D) harmonic oscillator [4], while their spectra and B(E2) transition rates are parameter-free (up to overall scale factors).

In Section 2 of the present paper we study a sequence of potentials interpolating between U(5) and E(5), while in Section 3 a new exactly soluble model, to be called X(5)-β^2, is introduced. A sequence of potentials interpolating between U(5) and X(5) is considered in Section 4, while the conclusions and plans for further work are given in Section 5.

2 Potentials interpolating between U(5) and E(5)

The original Bohr Hamiltonian [3] is

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}\right.$$

$$\left. -\frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}\right] + V(\beta,\gamma), \qquad (1)$$

where β and γ are the usual collective coordinates describing the shape of the nuclear surface, Q_k ($k = 1, 2, 3$) are the components of angular momentum, and B is the mass parameter.

Assuming that the potential depends only on the variable β, i.e. $V(\beta,\gamma) = U(\beta)$, one can proceed to separation of variables in the standard way [3,5], using the wavefunction $\Psi(\beta,\gamma,\theta_i) = f(\beta)\Phi(\gamma,\theta_i)$, where θ_i ($i = 1, 2, 3$) are the Euler angles describing the orientation of the deformed nucleus in space.

In the equation involving the angles, the eigenvalues of the second order Casimir operator of SO(5) occur, having the form $\Lambda = \tau(\tau+3)$, where $\tau = 0$, 1, 2, ... is the quantum number characterizing the irreducible representations (irreps) of SO(5), called the "seniority" [6]. This equation has been solved by Bes [7].

The "radial" equation can be simplified by introducing [1] reduced energies $\epsilon = \frac{2B}{\hbar^2}E$ and reduced potentials $u = \frac{2B}{\hbar^2}U$, as well as by making [1] the transformation $\phi(\beta) = \beta^{3/2}f(\beta)$, leading to the differential equation

$$\phi'' + \frac{\phi'}{\beta} + \left[\epsilon - u(\beta) - \frac{\left(\tau+\frac{3}{2}\right)^2}{\beta^2}\right]\phi = 0. \qquad (2)$$

For $u(\beta) = \beta^2/2$ one obtains the original solution of Bohr [3], which corresponds to a 5-dimensional (5-D) harmonic oscillator characterized by the symmetry U(5) $\supset$ SO(5) $\supset$ SO(3) $\supset$SO(2) [4], the eigenfunctions being proportional [8] to Laguerre polynomials $F_\nu^\tau(\beta) = \left[\frac{2\nu!}{\Gamma\left(\nu+\tau+\frac{5}{2}\right)}\right]^{\frac{1}{2}}\beta^\tau L_\nu^{\tau+\frac{3}{2}}(\beta^2)e^{-\beta^2/2}$,

where $\Gamma(n)$ stands for the Γ-function, and the spectrum having the simple form $E_N = N + \frac{5}{2}$, with $N = 2\nu + \tau$ and $\nu = 0, 1, 2, 3, \ldots$

For $u(\beta)$ being a 5-D infinite well ($u(\beta) = 0$ if $\beta \leq \beta_W$, while $u(\beta) = \infty$ for $\beta > \beta_W$), one obtains the E(5) model of Iachello [1], in which the eigenfunctions are Bessel functions $J_{\tau+3/2}(z)$ (with $z = \beta k$, $k = \sqrt{\epsilon}$), while the spectrum is given by $E_{\xi,\tau} = \frac{\hbar^2}{2B} k^2_{\xi,\tau}$, with $k_{\xi,\tau} = \frac{x_{\xi,\tau}}{\beta_W}$, where $x_{\xi,\tau}$ is the ξ-th zero of the Bessel function $J_{\tau+3/2}(z)$. The relevant symmetry in this case is E(5)$\supset$SO(5)$\supset$SO(3)$\supset$SO(2), where the Euclidean algebra in 5 dimensions, E(5), is generated by the 5-D momenta π_μ and the 5-D angular momenta $L_{\mu\nu}$, while SO(5) is generated by the $L_{\mu\nu}$ alone [2]. τ, L, and M are the quantum numbers characterizing the irreps of SO(5), SO(3), and SO(2) respectively. The values of angular momentum L contained in each irrep of SO(5) (i.e. for each value of τ) are given by the algorithm $\tau = 3\nu_\Delta + \lambda$, with $\nu_\Delta = 0, 1, \ldots$, and $L = \lambda, \lambda + 1, \ldots, 2\lambda - 2, 2\lambda$ (with $2\lambda - 1$ missing) [9], where ν_Δ is the missing quantum number in the reduction SO(5) $\supset$ SO(3).

The spectra of the $u(\beta) = \beta^2/2$ potential and of the E(5) model become directly comparable by establishing the formal correspondence $\nu = \xi - 1$, which allows one to continue using for the states the notation $L_{\xi,\tau}$ (where L is the angular momentum), as in Ref. [1], although a notation $L_{\nu,\tau}$ would have been equally appropriate. The ground state band corresponds to $\xi = 1$ (or, equivalently, $\nu = 0$).

The two cases mentioned above are the only ones in which Eq. (2) is exactly soluble, giving spectra characterized by $R_4 = E(4)/E(2)$ ratios 2.00 and 2.20 respectively. However, the numerical solution of Eq. (2) for potentials other than the ones mentioned above is a straightforward task [10], in which one uses the chain U(5)$\supset$SO(5)$\supset$SO(3)$\supset$SO(2) for the classification of the states.

Not all potentials can be used in Eq. (2), though, since they have to obey the restrictions imposed by the 24 transformations mentioned in [3] and listed explicitly in [11]. These restrictions allow the presence of even powers of β in the potentials, while odd powers of β should be accompanied by $\cos 3\gamma$ [12].

A particularly interesting sequence of potentials is given by $u_{2n}(\beta) = \frac{\beta^{2n}}{2}$, with n being an integer. For $n = 1$ the Bohr case (U(5)) is obtained, while for $n \to \infty$ the infinite well of E(5) is obtained [13]. Therefore this sequence of potentials provides a "bridge" between the U(5) symmetry and the E(5) model, using their common SO(5)$\supset$SO(3) chain of subalgebras for the classification of the spectra.

Numerical results for the spectra of the β^4, β^6, and β^8 potentials have been obtained through two different methods. In one approach, the representation of the position and momentum operators in matrix form [14] has been used,

while in the other the direct integration method [15] has been applied. In the latter, the differential equation is solved for each value of $\tau = 0, 1, 2, \ldots$ separately, the successive eigenvalues for each value of τ labeled by $\xi = 1, 2, 3, \ldots$ (or, equivalently, by $\nu = 0, 1, 2, \ldots$). The two methods give results mutually consistent, the second one appearing of more general applicability. Extensive tables of excitation energies relative to the ground state, normalized to the excitation energy of the first excited state, have been reported in Ref. [16].

For the above-mentioned potentials we introduce the labels E(5)-β^4, E(5)-β^6, E(5)-β^8, their meaning being that E(5)-β^{2n} corresponds to the potential $\beta^{2n}/2$ plugged in the differential equation obtained in the framework of the E(5) model. In this notation E(5)-β^2 coincides with the original U(5) model of Bohr [3], while E(5)-β^{2n} with $n \to \infty$ is simply the original E(5) model [1].

From the tables included in Ref. [16] it is clear that in all bands and for all values of the angular momentum, L, the potentials β^4, β^6, β^8 (which give R_4 ratios equal to 2.093, 2.135, and 2.157 respectively) gradually lead from the U(5) case ($R_4 = 2.000$) to the E(5) results ($R_4 = 2.199$) in a smooth way.

In nuclear structure it is well known that electromagnetic transition rates are quantities sensitive to the details of the underlying microscopic structure, as well as to details of the theoretical models, much more than the corresponding spectra. It is therefore a must to calculate B(E2) ratios (normalized to B(E2:$2_1^+ \to 0_1^+$)=100) for the potentials mentioned above.

The quadrupole operator has the form [5]

$$T^{(E2)}_{\mu} = t\alpha_{\mu} = t\beta \left[\mathcal{D}^{(2)}_{\mu,0}(\theta_i) \cos\gamma + \frac{1}{\sqrt{2}} (\mathcal{D}^{(2)}_{\mu,2}(\theta_i) + \mathcal{D}^{(2)}_{\mu,-2}(\theta_i)) \sin\gamma \right], \qquad (3)$$

where t is a scale factor and $\mathcal{D}(\theta_i)$ denote Wigner functions of the Euler angles, while the B(E2) transition rates are given by $B(E2; \varrho_i L_i \to \varrho_f L_f) = \frac{1}{2L_i+1} |\langle \varrho_f L_f || T^{(E2)} || \varrho_i L_i \rangle|^2$, where by ϱ quantum numbers other than the angular momentum L are denoted.

For the states with $\nu_\Delta = 0$ and $L = 2\tau$ one obtains

$$B(E2; (L+2)_{\xi',\tau+1} \to L_{\xi,\tau}) = \frac{\tau+1}{2\tau+5}\, t^2\, I^2_{\xi',\tau+1;\,\xi,\tau}, \qquad (4)$$

where $I_{\xi',\tau+1;\,\xi,\tau} = \int_0^\infty \beta f_{\xi'\,\tau+1}(\beta)\, f_{\xi\tau}(\beta) \beta^4 d\beta$.

In the special case of the potential being a 5-D infinite well the eigenfunctions are $f_{\xi\tau}(\beta) = \frac{1}{\sqrt{C_{\xi,\tau}}}\, \beta^{-3/2} J_{\tau+3/2}\Big(x_{\xi,\tau}\, \frac{\beta}{\beta_W}\Big)$, with $C_{\xi,\tau} = \frac{\beta_W^2}{2} J^2_{\tau+5/2}(x_{\xi,\tau})$, where $x_{\xi,\tau}$ is the ξ-th zero of the Bessel function $J_{\tau+3/2}(z)$, while the constants $C_{\xi,\tau}$ are obtained from the normalization condition $\int_0^{\beta_W} f^2_{\xi\tau}(\beta)\ \beta^4 d\beta = 1$. In this

case the relevant integrals take the form

$$I_{\xi',\tau+1;\,\xi,\tau} = (C_{\xi',\tau+1}C_{\xi,\tau})^{-1/2}\beta_W^3 \int_0^1 z^2 J_{\tau+5/2}(x_{\xi',\tau+1}\,z)\,J_{\tau+3/2}(x_{\xi,\tau}\,z)\,dz. \quad (5)$$

Extensive results of the calculations for intraband and interband transitions have been reported in Ref. [16]. In all cases a smooth evolution from U(5) to E(5) is seen. The E(5) results reported in Ref. [16] are in good agreement with the results given in Ref. [17].

It is interesting to examine if there is any experimental evidence supporting the E(5)-β^{2n} predictions. It is clear that the first regions to be considered are the ones around the nuclei which have been identified as good candidates for E(5), i.e. ^{134}Ba [18], ^{104}Ru [19], ^{102}Pd [20]. A very preliminary search indicates that ^{98}Ru can be a candidate for E(5)-β^6, while ^{100}Pd can be a candidate for E(5)-β^4. However, much more detailed information on the spectra and B(E2) transitions of these nuclei is required before final conclusions can be reached.

3 X(5)-β^2: A new exactly soluble model

The starting point is again the original Bohr Hamiltonian [3] of Eq. (1). One seeks solutions of the relevant Schrödinger equation having the form $\Psi(\beta,\gamma,\theta_i) = \phi_K^L(\beta,\gamma)\mathcal{D}_{M,K}^L(\theta_i)$, where θ_i (i = 1, 2, 3) are the Euler angles, $\mathcal{D}(\theta_i)$ denote Wigner functions of them, L are the eigenvalues of angular momentum, while M and K are the eigenvalues of the projections of angular momentum on the laboratory-fixed z-axis and the body-fixed z'-axis respectively.

As pointed out in Ref. [2], in the case in which the potential has a minimum around $\gamma = 0$ one can write the last term of the Bohr Hamiltonian of Eq. (1) in the form given in Eq. (2) of Ref. [2]. Using this result in the Schrödinger equation corresponding to the original Bohr Hamiltonian, introducing reduced energies $\epsilon = 2BE/\hbar^2$ and reduced potentials $u = 2BV/\hbar^2$, and assuming that the reduced potential can be separated into two terms, one depending on β and the other depending on γ, i.e. $u(\beta,\gamma) = u(\beta) + u(\gamma)$, the Schrödinger equation can be separated [2] into two equations

$$\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{4\beta^2}\frac{4}{3}L(L+1) + u(\beta)\right]\xi_L(\beta) = \epsilon_\beta\xi_L(\beta), \quad (6)$$

$$\left[-\frac{1}{\langle\beta^2\rangle\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4\langle\beta^2\rangle}K^2\left(\frac{1}{\sin^2\gamma} - \frac{4}{3}\right) + u(\gamma)\right]\eta_K(\gamma)$$

$$= \epsilon(\gamma)\eta_K(\gamma), \tag{7}$$

where $\langle\beta^2\rangle$ is the average of β^2 over $\xi(\beta)$ and $\epsilon = \epsilon_\beta + \epsilon_\gamma$.

In Ref. [2] Eq. (6) is solved exactly for the case in which $u(\beta)$ is an infinite well potential, as the one used in Section 2. The relevant exactly soluble model is labeled as X(5) (which is not meant as a group label, although there is relation to projective representations of E(5), the Euclidean group in 5 dimensions [2]). In particular, Eq. (6) in the case of $u(\beta)$ being an infinite well potential is transformed into a Bessel equation, the relevant eigenvalues being $\epsilon_{\beta;s,L} = (k_{s,L})^2$, with $k_{s,L} = \frac{x_{s,L}}{\beta_W}$, where $x_{s,L}$ is the s-th zero of the Bessel function $J_\nu(k_{s,L}\beta)$, with $\nu = \left(\frac{L(L+1)}{3} + \frac{9}{4}\right)^{1/2}$, while the relevant eigenfunctions are $\xi_{s,L}(\beta) = c_{s,L}\beta^{-3/2}J_\nu(k_{s,L}\beta)$, where $c_{s,L}$ are normalization constants.

Eq. (6) is exactly soluble also in the case in which $u(\beta) = \beta^2/2$. In this case, to which we are going to refer as the X(5)-β^2 model, the eigenfunctions [8] are

$$F_n^L(\beta) = \left[\frac{2n!}{\Gamma\left(n + a + \frac{5}{2}\right)}\right]^{1/2} \beta^a L_n^{a+\frac{3}{2}}(\beta^2)e^{-\beta^2/2}, \tag{8}$$

where $\Gamma(n)$ stands for the Γ-function, $L_n^a(z)$ denotes the Laguerre polynomials, and $a = \frac{1}{2}\left(-3 + \sqrt{9 + \frac{4}{3}L(L+1)}\right)$, while the energy eigenvalues are $E_{n,L} = 2n + 1 + \sqrt{\frac{9}{4} + \frac{L(L+1)}{3}}$, with $n = 0, 1, 2, \ldots$

In the above, n is the usual oscillator quantum number. One can see that a formal correspondence between the energy levels of the X(5) model and the present X(5)-β^2 model, can be established through the relation $n = s - 1$. In the present notation, the ground state band corresponds to $s = 1$ ($n = 0$). For the energy states the notation $E_{s,L} = E_{n+1,L}$ of Ref. [2] will be kept.

In the original version of the X(5) model [2] the potential $u(\gamma)$ is considered as a harmonic oscillator potential. The energy eigenvalues turn out to be $E(s, L, n_\gamma, K, M) = E_0 + B(x_{s,L})^2 + An_\gamma + CK^2$, where n_γ and K obtain the values $n_\gamma = 0$, $K = 0$; $n_\gamma = 1$, $K = \pm 2$; $n_\gamma = 2$, $K = 0, \pm 4$; ... For $K = 0$ one has $L = 0$, 2, 4, ..., while for $K \neq 0$ one obtains $L = K$, $K + 1$, $K + 2$, ...

In the present X(5)-β^2 model, one also uses for $u(\gamma)$ a harmonic oscillator potential, as in the X(5) model. As a consequence, the full spectrum is given by $E(n, L, n_\gamma, K, M) = E_0' + B'E_{n,l} + A'n_\gamma + C'K^2$, which is an analogue of the equation given above in the case of X(5). The sets of values obtained by n_γ, K, and L remain unchanged.

Extensive numerical results for the β-parts of the energy spectra (which correspond to no excitations in the γ-variable, i.e. to $n_\gamma = 0$) of the X(5)-β^2 and X(5) models have been reported in Ref. [21]. All levels are normalized

to the energy of the first excited state, $E_{1,2} - E_{1,0} = 1.0$, where the notation $E_{s,L} = E_{n+1,L}$ is used. The model predictions for these bands are parameter independent, up to an overall scale. This is not the case for bands with $n_\gamma \neq 0$, since in this case, as seen above, the extra parameters A, C and A', C' enter respectively. Therefore, in the case of the ($n_\gamma = 1$, $K = 2$)-band, the energies are listed in Ref. [21] after subtracting from them the relevant $L = 2$ bandhead, using the same normalization as above. The R_4 ratio turns out to be 2.646 .

The quadrupole operator has the form of Eq. (3) [5], while the B(E2) transition rates are again given by the equation following it. The matrix elements of the quadrupole operator involve an integral over the Euler angles, which is the same as in Ref. [2] and is performed by using the properties of the Wigner $\mathcal{D}$ functions, of which only $\mathcal{D}^{(2)}_{\mu,0}$ participates, since $\gamma \simeq 0$ in Eq. (3) (as mentioned in the beginning of the present section), as well as an integral over β. After performing the integrations over the angles one is left with $B(E2; L_s \rightarrow L'_{s'}) = (L_s 2 L'_{s'}|000)^2 I^2_{s,L;s',L'}$, where the Clebsch–Gordan coefficient $(L_s 2 L'_{s'}|000)$ appears, which determines the relevant selection rules. In the case of X(5) the integral over β is $I_{s,L;s',L'} = \int \beta \xi_{s,L}(\beta)\xi_{s',L'}(\beta)\beta^4 d\beta$, which involves Bessel functions, while in the case of X(5)-β^2 the integral has the form $I_{s,L;s',L'} = \int \beta F^L_n(\beta) F^{L'}_{n'} \beta^4 d\beta$, with $n = s - 1$ and $n' = s' - 1$, which involves Laguerre polynomials.

Extensive results for intraband and interband transitions have been reported in Ref. [21]. All transitions are normalized to $B(E2 : 2^+_1 \rightarrow 0^+_1) = 100$.

4 A sequence of potentials lying between U(5) and X(5)

The two cases mentioned in the previous section are the only ones in which Eq. (6) is exactly soluble, giving spectra characterized by R_4 ratios 2.646 and 2.904 for X(5)-β^2 and X(5) respectively. However, the numerical solution of Eq. (6) for other potentials is a straightforward task. The potentials to be used in Eq. (6) have to obey the restrictions imposed by the 24 transformations mentioned in [3] and listed explicitly in [11].

A particularly interesting sequence of potentials is given by $u_{2n}(\beta) = \frac{\beta^{2n}}{2}$, with n being an integer. For $n = 1$ the X(5)-β^2 case is obtained, while for $n \rightarrow \infty$ the infinite well of X(5) is obtained [13]. Therefore this sequence of potentials interpolates between the X(5)-β^2 model and the X(5) model, in the region lying between U(5) and X(5).

Numerical results for the spectra of the β^4, β^6, and β^8 potentials have been obtained through the two different methods described in Section 2. Extensive

results have been reported in Ref. [21], where excitation energies relative to the ground state, normalized to the excitation energy of the first excited state, are exhibited.

For the above-mentioned potentials the model labels X(5)-β^4, X(5)-β^6, X(5)-β^8 are introduced, their meaning being that the X(5)-β^{2n} model corresponds to the potential $\beta^{2n}/2$ plugged in the differential equation of Eq. (6) obtained in the framework of the X(5) model. In this notation X(5)-β^{2n} with $n \to \infty$ is simply the original X(5) model [2].

From the results reported in Ref. [21] it is clear that in all bands and for all values of the angular momentum, L, the potentials β^4, β^6, β^8 (which give R_4 ratios equal to 2.769, 2.824, and 2.852 respectively) gradually lead from the X(5)-β^2 case ($R_4 = 2.646$) to the X(5) results ($R_4 = 2.904$) in a smooth way.

The calculation of the B(E2)s follows the steps described in the end of Section 3. The same general equation is still valid, the only difference being that in the integral over β the wave functions in the present cases are known only in numerical form and not in analytic form as in the X(5) and X(5)-β^2 cases.

Extensive results of the calculations for intraband and interband transitions have also been reported in Ref. [21]. In all cases a smooth evolution from X(5)-β^2 to X(5) is seen.

It is clear that the first place to look for nuclei exhibiting X(5)-β^{2n} behaviour is the region close to nuclei showing X(5) structure. The best examples of nuclei corresponding to the X(5) structure are so far the $N = 90$ isotones ^{152}Sm [22], ^{150}Nd [23], ^{156}Dy [24]. A preliminary search in the rare earths with $N < 90$ shows that ^{148}Nd can be a candidate for X(5)-β^2, ^{158}Er can be a candidate for X(5)-β^6, while ^{160}Yb can be a candidate for X(5)-β^4. However, much more detailed information on spectra and B(E2) transitions is needed before final conclusions can be reached.

5 Conclusion

It has been proved that the potentials β^{2n} (with n being integer) provide a complete "bridge" between the U(5) symmetry of the Bohr Hamiltonian with a harmonic oscillator potential (occuring for $n = 1$) and the E(5) model of Iachello, which is obtained from the Bohr Hamiltonian when an infinite well potential is plugged in it (materialized for $n \to \infty$). Parameter-free (up to overall scale factors) predictions for spectra and B(E2) transition rates have been given for the potentials β^4, β^6, β^8, called the E(5)-β^4, E(5)-β^6, and E(5)-β^8 models, respectively. Hints about nuclei showing this behaviour have been

briefly discussed.

In addition, an exactly soluble model, labeled as X(5)-β^2, has been constructed starting from the original Bohr collective Hamiltonian, separating the β and γ variables as in the X(5) model of Iachello, and using a harmonic oscillator potential for the β-variable. Furthermore it has been proved that the potentials β^{2n} (with n being integer) provide a "bridge" between this new X(5)-β^2 model (occuring for $n = 1$) and the X(5) model of Iachello (which is obtained by putting in the Bohr Hamiltonian an infinite well potential in the β-variable, materialized for $n \to \infty$). Parameter-free (up to overall scale factors) predictions for spectra and B(E2) transition rates have been given for the potentials β^2, β^4, β^6, β^8, called the X(5)-β^2, X(5)-β^4, X(5)-β^6, and X(5)-β^8 models, respectively, lying between the U(5) symmetry of the original Bohr Hamiltonian and the X(5) model. Hints about nuclei showing this behaviour have been given.

Concerning future theoretical work, one should try to find a sequence of potentials interpolating between O(6) and E(5), as well as between SU(3) and X(5). In other words, one should try to approach E(5) and X(5) "from the other side". From the classical limit of the O(6) and SU(3) symmetries of the Interacting Boson Model [9] it is clear that for this purpose potentials with a minimum at $\beta \neq 0$ should be considered, the potentials $u_{2n}^D(\beta) = \beta^{2n} + \frac{\beta_0^{4n}}{\beta^{2n}}$ being strong candidates. The Davidson potential, corresponding to $n = 1$, is known to be exactly soluble [25,26].

Work in these directions is in progress.

Acknowledgements

The authors are thankful to Rick Casten (Yale), Jean Libert (Orsay), and Werner Scheid (Giessen) for illuminating discussions. Support through the NATO Collaborative Linkage Grant PST.CLG 978799 is gratefully acknowledged.

References

[1] F. Iachello, Phys. Rev. Lett. 85 (2000) 3580.

[2] F. Iachello, Phys. Rev. Lett. 87 (2001) 052502.

[3] A. Bohr, Mat. Fys. Medd. K. Dan. Vidensk. Selsk. 26 (1952) no. 14.

[4] E. Chacón and M. Moshinsky, J. Math. Phys. 18 (1977) 870.

[5] L. Wilets and M. Jean, Phys. Rev. 102 (1956) 788.

[6] G. Rakavy, Nucl. Phys. 4 (1957) 289.

[7] D. R. Bes, Nucl. Phys. 10 (1959) 373.

[8] M. Moshinsky, J. Math. Phys. 25 (1984) 1555.

[9] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge Univesrity Press, Cambridge, 1987).

[10] D. Troltenier, J. A. Maruhn, and P. O. Hess, in *Computational Nuclear Physics 1. Nuclear Structure*, ed. K. Langanke, J. A. Maruhn, and S. E. Koonin (Springer, Berlin, 1991) p. 105.

[11] T. M. Corrigan, F. J. Margetan, and S. A. Williams, Phys. Rev. C 14 (1976) 2279.

[12] W. Greiner and J. A. Maruhn, *Nuclear Models* (Springer, Berlin, 1996).

[13] C. M. Bender, S. Boettcher, H. F. Jones, and V. M. Savage, J. Phys. A 32 (1999) 6771.

[14] H. J. Korsch and M. Glück, Eur. J. Phys. 23 (2002) 413.

[15] N. Minkov and W. Scheid, INRNE Sofia preprint (2003).

[16] D. Bonatsos, D. Lenis, N. Minkov, P. P. Raychev and P. A. Terziev, N.C.S.R. Demokritos preprint DEM-NT-03-01, to appear in the *Proceedings of the XXII International Workshop on Nuclear Theory (Rila Mountains, Bulgaria, 2003)*, ed. V. Nikolaev.

[17] F. Iachello, in *Mapping the Triangle: International Conference on Nuclear Structure*, ed. A. Aprahamian, J. A. Cizewski, S. Pittel, and N. V. Zamfir (American Institute of Physics, 2002), Vol. CP638, p. 1.

[18] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. 85 (2000) 3584.

[19] A. Frank, C. E. Alonso, and J. M. Arias, Phys. Rev. C 65 (2001) 014301.

[20] N. V. Zamfir *et al.*, Phys. Rev. C 65 (2002) 044325.

[21] D. Bonatsos, D. Lenis, N. Minkov, P. P. Raychev and P. A. Terziev, N.C.S.R. Demokritos preprint DEM-NT-03-02, to appear in the *Proceedings of the XXII International Workshop on Nuclear Theory (Rila Mountains, Bulgaria, 2003)*, ed. V. Nikolaev.

[22] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. 87 (2001) 052503.

[23] R. Krücken *et al.*, Phys. Rev. Lett. 88 (2002) 232501.

[24] M. A. Caprio *et al.*, Phys. Rev. C 66 (2002) 054310.

[25] P. M. Davidson, Proc. R. Soc. 135 (1932) 459.

[26] D. J. Rowe and C. Bahri, J. Phys. A 31 (1998) 4947.