**Variational procedure leading from Davidson potentials to the E(5)and X(5) critical point symmetries**

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# Variational procedure leading from Davidson potentials to the E(5)and X(5) critical point symmetries

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## Abstract

Davidson potentials of the form $\beta^2 + \beta_0^4/\beta^2$, when used in the original Bohr Hamiltonian for $\gamma$-independent potentials bridge the U(5) and O(6) symmetries. Using a variational procedure, we determine for each value of angular momentum $L$ the value of $\beta_0$ at which the derivative of the energy ratio $R_L = E(L)/E(2)$ with respect to $\beta_0$ has a sharp maximum, the collection of $R_L$ values at these points forming a band which practically coincides with the ground state band of the E(5) model, corresponding to the critical point in the shape phase transition from U(5) to O(6). The same potentials, when used in the Bohr Hamiltonian after separating variables as in the X(5) model, bridge the U(5) and SU(3) symmetries, the same variational procedure leading to a band which practically coincides with the ground state band of the X(5) model, corresponding to the critical point of the U(5) to SU(3) shape phase transition. A new derivation of the Holmberg–Lipas formula for nuclear energy spectra is obtained as a by-product.

*Key words:* Davidson potential, Shape phase transition, Variational method, E(5) model, X(5) model
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## 1 Introduction

The recently introduced E(5) [1] and X(5) [2] models are supposed to describe shape phase transitions in atomic nuclei, the former being related to

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[1] Deceased

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the transition from U(5) (vibrational) to O(6) ($\gamma$-unstable) nuclei, and the latter corresponding to the transition from U(5) to SU(3) (rotational) nuclei. In both cases the original Bohr collective Hamiltonian [3] is used, with an infinite well potential in the collective $\beta$-variable. Separation of variables is achieved in the E(5) case by assuming that the potential is independent of the collective $\gamma$-variable, while in the X(5) case the potential is assumed to be of the form $u(\beta) + u(\gamma)$. We are going to refer to these two cases as "the E(5) framework" and "the X(5) framework" respectively. In the present work we examine if the choice of the infinite well potential is the optimum one for the description of shape phase transitions. For this purpose, we need one-parameter potentials which can span the U(5)-O(6) region in the E(5) framework, as well as the U(5)-SU(3) region in the X(5) framework. It turns out that the exactly soluble [4,5] Davidson potentials [6]

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2},$$ (1)

where $\beta_0$ is the position of the minimum of the potential, do possess this property. Taking into account the fact that various physical quantities should change most rapidly at the point of the shape phase transition [7], we locate for each value of the angular momentum $L$ the value of $\beta_0$ for which the rate of change of the ratio $R_L = E(L)/E(2)$, a widely used measure of nuclear collectivity, is maximized. It turns out that the collection of $R_L$ ratios formed in this way in the case of a potential independent of the $\gamma$-variable correspond to the E(5) model, while in the case of the $u(\beta) + u(\gamma)$ potential lead to the X(5) model, thus proving that the choice of the infinite well potential made in Refs. [1,2] is the optimum one.

## 2 Davidson potentials in the E(5) framework

The original Bohr Hamiltonian [3] is

$$H = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} \right.$$
$$\left. - \frac{1}{4\beta^2} \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2 \left( \gamma - \frac{2}{3}\pi k \right)} \right] + V(\beta, \gamma),$$ (2)

where $\beta$ and $\gamma$ are the usual collective coordinates describing the shape of the nuclear surface, $Q_k$ ($k = 1, 2, 3$) are the components of angular momentum, and $B$ is the mass parameter.

Assuming that the potential depends only on the variable $\beta$, i.e. $V(\beta, \gamma) = U(\beta)$, one can proceed to separation of variables in the standard way [3,8], using the wave function $\Psi(\beta, \gamma, \theta_i) = f(\beta)\Phi(\gamma, \theta_i)$, where $\theta_i$ $(i = 1, 2, 3)$ are the Euler angles describing the orientation of the deformed nucleus in space. In the equation involving the angles, the eigenvalues of the second order Casimir operator of SO(5) occur, having the form $\Lambda = \tau(\tau+3)$, where $\tau = 0, 1, 2, \ldots$ is the quantum number characterizing the irreducible representations (irreps) of SO(5), called the "seniority" [9]. This equation has been solved by Bes [10]. The "radial" equation can be simplified by introducing [1] reduced energies $\epsilon = \frac{2B}{\hbar^2}E$ and reduced potentials $u = \frac{2B}{\hbar^2}U$, leading to

$$\left[ -\frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{\tau(\tau + 3)}{\beta^2} + u(\beta) \right] f(\beta) = \epsilon f(\beta). \tag{3}$$

When plugging the Davidson potentials of Eq. (1) in the above equation, the $\beta_0^4/\beta^2$ term is combined with the $\tau(\tau + 3)/\beta^2$ term appearing there and the equation is solved exactly [4,5], the eigenfunctions being Laguerre polynomials, while the energy eigenvalues are given by [4,5] (in $\hbar\omega = 1$ units)

$$E_{n,\tau} = 2n + 1 + \left[ \left( \tau + \frac{3}{2} \right)^2 + \beta_0^4 \right]^{1/2}. \tag{4}$$

The levels of the ground state band are characterized by $L = 2\tau$ and $n = 0$. Then the energy levels of the ground state band are given by

$$E_{0,L} = 1 + \frac{1}{2} \left[ (L + 3)^2 + 4\beta_0^4 \right]^{1/2}, \tag{5}$$

while the excitation energies of the levels of the ground state band relative to the ground state are

$$E_{0,L,exc} = E_{0,L} - E_{0,0} = \frac{1}{2} \left( \left[ (L + 3)^2 + 4\beta_0^4 \right]^{1/2} - \left[ 9 + 4\beta_0^4 \right]^{1/2} \right). \tag{6}$$

For $u(\beta)$ being a 5-D infinite well one obtains the E(5) model of Iachello [1] in which the eigenfunctions are Bessel functions $J_{\tau+3/2}(z)$ (with $z = \beta k$, $k = \sqrt{\epsilon}$), while the spectrum is determined by the zeros of the Bessel functions.

It is instructive to consider the ratios

$$R_L = \frac{E_{0,L} - E_{0,0}}{E_{0,2} - E_{0,0}}, \tag{7}$$

where the notation $E_{n,L}$ is used. For $\beta_0 = 0$ it is clear that the original vibrational model of Bohr [3] (with $R_4 = 2$) is obtained, while for large $\beta_0$ the O(6)

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Table 1

Parameter values $\beta_{0,max}$ where the first derivative of the energy ratios $R_L$ (defined in Eq. (7)) in the E(5) framework has a maximum, while the second derivative vanishes, together with the $R_L$ ratios obtained at these values (labeled by "var") and the corresponding ratios of the E(5) model, for several values of the angular momentum $L$.

| L | $\beta_{0,max}$ | $R_L$ var | $R_L$ E(5) |
|---|---|---|---|
| 4 | 1.421 | 2.185 | 2.199 |
| 6 | 1.522 | 3.549 | 3.590 |
| 8 | 1.609 | 5.086 | 5.169 |
| 10 | 1.687 | 6.793 | 6.934 |
| 12 | 1.759 | 8.667 | 8.881 |
| 14 | 1.825 | 10.705 | 11.009 |
| 16 | 1.888 | 12.906 | 13.316 |
| 18 | 1.947 | 15.269 | 15.799 |
| 20 | 2.004 | 17.793 | 18.459 |

limit of the Interacting Boson Model (IBM) [12] (with $R_4 = 2.5$) is approached [4]. One can easily see that these ratios increase with $\beta_0$, the increase becoming very steep at some value $\beta_{0,max}$ of $\beta_0$, where the first derivative $\frac{dR_L}{d\beta_0}$ reaches a maximum value, while the second derivative $\frac{d^2 R_L}{d\beta_0^2}$ vanishes. Then the values of $\beta_0$ at which the first derivative $dR_L/d\beta_0$ exhibits a sharp maximum are determined for each value of the angular momentum $L$ separately, the collection of $R_L$ ratios at these values of $\beta_0$ forming a band, which turns out to be in very good agreement with the ground state band of E(5), the model supposed to be appropriate for describing nuclei at the critical point in the transition from U(5) to O(6), thus indicating that the choice of the infinite well potential used in the E(5) model is the optimum one. The results are depicted in Table 1.

## 3  Davidson potentials in the X(5) framework

Starting again from the original Bohr Hamiltonian [3], one seeks solutions of the relevant Schrödinger equation of the form $\Psi(\beta, \gamma, \theta_i) = \phi_K^L(\beta, \gamma) D_{M,K}^L(\theta_i)$, where $\theta_i$ ($i = 1, 2, 3$) are the Euler angles, $D(\theta_i)$ denote Wigner functions of them, $L$ are the eigenvalues of angular momentum, while $M$ and $K$ are the eigenvalues of the projections of angular momentum on the laboratory-fixed $z$-axis and the body-fixed $z'$-axis respectively. As pointed out in Ref. [2], an approximate separation of variables can be achieved by assuming that

the potential has a minimum around $\gamma = 0$, as well as that it can be separated into two terms, one depending on $\beta$ and the other depending on $\gamma$, i.e. $u(\beta, \gamma) = u(\beta) + u(\gamma)$. As described in more detail in Ref. [11], the "radial" equation occuring in this case can be solved exactly in the case of the Davidson potentials of Eq. (1), the energy eigenvalues being (in $\hbar\omega = 1$ units)

$$E_{n,L} = 2n + 1 + \left[ \frac{1}{3}L(L+1) + \frac{9}{4} + \beta_0^4 \right]^{1/2} \tag{8}$$

The levels of the ground state band are characterized by $n = 0$. With appropriate normalization we get the expression

$$E'_{0,L,exc} = \frac{E_{0,L,exc}}{\left[ \frac{9}{4} + \beta_0^4 \right]^{1/2}} = \left[ 1 + \frac{L(L+1)}{3\left( \frac{9}{4} + \beta_0^4 \right)} \right]^{1/2} - 1, \tag{9}$$

which is the same as the Holmberg–Lipas formula [13]

$$E_H(L) = a_H \left( \sqrt{1 + b_H L(L+1)} - 1 \right), \tag{10}$$

with $a_H = 1$ and $b_H = 1/(27/4 + 3\beta_0^4)$.

It is clear that the Holmberg–Lipas formula gives rotational spectra for small values of $b_H$, at which one can keep only the first $L$-dependent term in the Taylor expansion of the square root appearing in Eq. (10), leading to energies proportional to $L(L+1)$. From the expression for $b_H$ it is then clear that rotational spectra are expected for large values of $\beta_0$. On the other hand, the case $\beta_0 = 0$ corresponds to an exactly soluble model with $R_4 = 2.646$, which has been called the X(5)-$\beta^2$ model [14]. Following the same variational procedure as before, we produce a collection of $R_L$ ratios in very good agreement with the ground state band of the X(5) model, as is shown in Table 2.


## 4   Discussion


A variational procedure for determining the values of physical quantities at the point of shape phase transitions in nuclei has been suggested. Using one-parameter potentials spanning the region between the two limiting symmetries of interest, the parameter values at which the rate of change of the physical quantity becomes maximum are determined for each value of the angular momentum separately and the corresponding values of the physical quantity at these parameter values are calculated.

Table 2
Parameter values $\beta_{0,max}$ where the first derivative of the energy ratios $R_L$ (defined in Eq. (7)) in the X(5) framework has a maximum, while the second derivative vanishes, together with the $R_L$ ratios obtained at these values (labeled by "var") and the corresponding ratios of the X(5) model, for several values of the angular momentum $L$.

| L | $\beta_{0,max}$ | $R_L$ var | $R_L$ X(5) |
|---|---|---|---|
| 4 | 1.334 | 2.901 | 2.904 |
| 6 | 1.445 | 5.419 | 5.430 |
| 8 | 1.543 | 8.454 | 8.483 |
| 10 | 1.631 | 11.964 | 12.027 |
| 12 | 1.711 | 15.926 | 16.041 |
| 14 | 1.785 | 20.330 | 20.514 |
| 16 | 1.855 | 25.170 | 25.437 |
| 18 | 1.922 | 30.442 | 30.804 |
| 20 | 1.985 | 36.146 | 36.611 |

The method has been applied in the shape phase transition from U(5) to O(6), using one-parameter Davidson potentials [6] and considering the energy ratios $R_L = E(L)/E(2)$ within the ground state band as the relevant physical quantity, leading to a band which practically coincides with the ground state band of the E(5) model [1]. It has also been applied in the same way in the shape phase transition from U(5) to SU(3), leading to a band which practically coincides with the ground state band of the X(5) model [2].

It should be emphasized that the application of the method was possible because the Davidson potentials correctly reproduce the U(5) and O(6) symmetries in the former case (for small and large parameter values respectively), as well as the relevant X(5)-$\beta^2$ [14] and SU(3) symmetries in the latter case (for small and large parameter values respectively).

As a by-product, a derivation of the Holmberg–Lipas formula [13] has been achieved using Davidson potentials in the X(5) framework.

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