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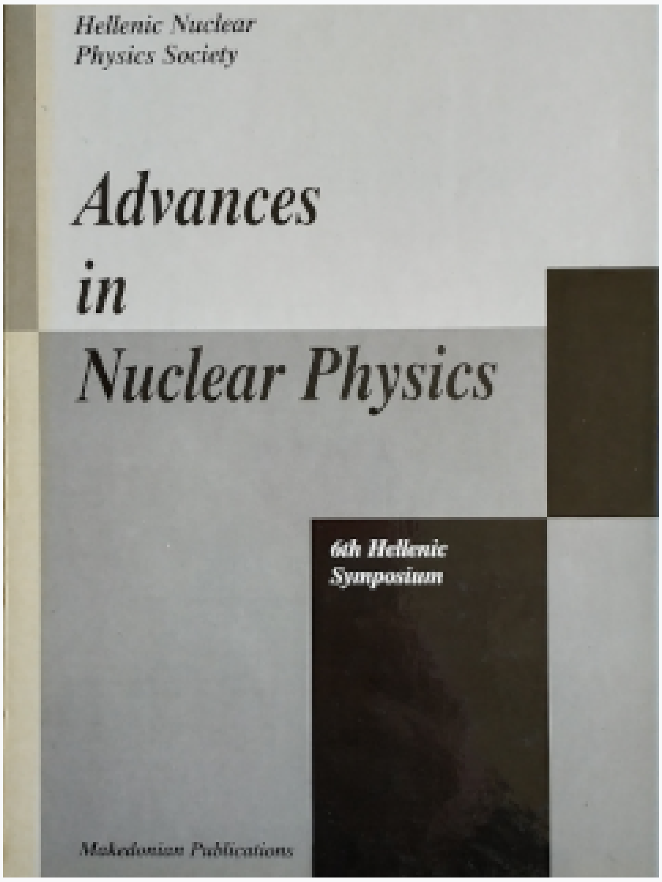

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Quasi-Exactly Soluble Potentials and Deformed Oscillators

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Abstract

It is proved that quasi-exactly soluble potentials corresponding to an oscillator with harmonic, quartic and sextic terms, for which the $n+1$ lowest levels of a given parity can be determined exactly, may be approximated by WKB equivalent potentials corresponding to deformed anharmonic oscillators of $\mathrm{SU}_q(1,1)$ symmetry, which have been used for the description of vibrational spectra of diatomic molecules. This connection allows for the immediate approximate determination of the levels of the same parity lying above the lowest $n+1$ known levels, as well as of all levels of the opposite parity. Such connections are not possible in the cases of the q-deformed oscillator, the Q-deformed oscillator, and the modified Pöschl-Teller potential with SU(1,1) symmetry.

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1 Introduction

Quantum algebras (also called quantum groups), which are deformed versions of the usual Lie algebras, to which they reduce when the deformation parameter is set equal to unity, have recently been attracting considerable attention. The interest in possible physical applications was triggered by the introduction of the q-deformed harmonic oscillator by Biedenharn and Macfarlane in 1989, although similar oscillators had already been in existence. The common mathematical structure of these various types of oscillators was shown by several authors (see [1] for a list of references), using the concept of the generalized deformed oscillator.

One way to clarify the physical content of deformed oscillators is to try to construct potentials which give, exactly or approximately, the same spectrum as these oscillators. Along these lines, WKB equivalent potentials corresponding to the q-deformed oscillator, as well as to deformed anharmonic oscillators found useful in the description of vibrational spectra of diatomic molecules [2] have been constructed, both numerically and in analytic form [3]. These potentials give approximately the same spectrum as the deformed oscillators under discussion.

On the other hand, several quasi-exactly soluble potentials (QESPs) have been introduced [4, 5, 6], for which the exact calculation of the first $n+1$ energy levels is possible, with no information provided for the rest of the levels. In particular, in the case of the harmonic oscillator with both quartic and sextic anharmonicities [4], the exact calculation of the first $n+1$ energy levels *of a given parity* is possible.

It is therefore interesting to check if there is any relation between the WKB equivalent potentials (WKB-EPs) approximating the behavior of the deformed oscillators on the one hand and the quasi-exactly soluble potentials on the other. If such a link exists, as we shall demonstrate in this letter, one can fix the parameters of a WKB potential equivalent to an appropriate oscillator so that the potential agrees, up to the sextic term, with a given quasi-exactly soluble potential. If the approximations involved (WKB method, truncation of the WKB potential to sixth order only) are good, one can subsequently use the levels of the relevant oscillator in order to approximate the unknown higher levels of the quasi-exactly soluble potential. In the case of the harmonic oscillator with both quartic and sextic anharmonicities [4], the levels of the opposite parity can also be approximated in this way.

2 Deformed oscillators

The q-numbers are defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{1}$$

In the case that $q = e^\tau$, where τ real, they can be written as

$$[x] = \frac{\sinh(\tau x)}{\sinh(\tau)}, \tag{2}$$

while in the case that q is a phase ($q = e^{i\tau}$, with τ real) they take the form

$$[x] = \frac{\sin(\tau x)}{\sin(\tau)}. \tag{3}$$

It is clear that in both cases the q-numbers reduce to the usual numbers as $q \to 1$ ($\tau \to 0$).

The q-deformed harmonic oscillator is determined by the creation and annihilation operators a^+ and a, which satisfy the relations:

$$aa^+ - q^{\mp 1} a^+ a = q^{\pm N}, \quad [N, a^+] = a^+, \quad [N, a] = -a, \tag{4}$$

where N is the number operator. The Hamiltonian of the q-deformed oscillator is

$$H = \frac{\hbar\omega}{2}(aa^+ + a^+ a), \tag{5}$$

and its eigenvalues are

$$E(n) = \frac{\hbar\omega}{2}([n] + [n+1]). \tag{6}$$

From Eq. (6) it is clear that the spectrum of the q-deformed harmonic oscillator is not equidistant. For real q the spectrum increases more rapidly than in the classical case, while for q complex it increases less rapidly than in the classical case, i.e. it gets squeezed.

In addition to the q-numbers, Q-numbers have been defined in the literature as

$$[x]_Q = \frac{Q^x - 1}{Q - 1}. \tag{7}$$

The corresponding Q-deformed harmonic oscillator is determined by the creation and annihilation operators b^+ and b, satisfying the commutation relations

$$bb^+ - Qb^+ b = 1, \quad [N, b^+] = b^+, \quad [N, b] = -b. \tag{8}$$

The Hamiltonian of the relevant oscillator is

$$H = \frac{\hbar\omega}{2}(bb^+ + b^+ b), \tag{9}$$

with corresponding eigenvalues

$$E(n) = \frac{\hbar\omega}{2}([n]_Q + [n+1]_Q). \tag{10}$$

If $Q = e^T$, with T real, the spectrum increases more rapidly than an equidistant spectrum for $T > 0$, while it increases less rapidly than an equidistant spectrum for $T < 0$.

Using two q-deformed oscillators described by the operators a_1, a_1^+, N_1, and a_2, a_2^+, N_2, the generators of the quantum algebra $\mathrm{SU}_q(1,1)$ are written as

$$K_+ = a_1^+ a_2^+, \quad K_{=} a_1 a_2, \quad K_0 = \frac{1}{2}\left(N_1 + N_2 + 1\right), \tag{11}$$

and satisfy the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -\left[2K_0\right], \tag{12}$$

where the square bracket in the rhs of the last equation is a q-number as defined in eq. (1). In ref. [2] this symmetry has been used for the description of vibrational molecular spectra. The q-deformed anharmonic oscillator used there is a q-generalization of the anharmonic oscillators used in the usual Lie algebraic approach to molecular spectroscopy. (See ref. [2] for detailed references.)

3 WKB equivalent potentials

In the case in which $q = e^{i\tau}$, the spectrum of the q-oscillator (Eq. (6)) can be written as

$$E_n = \frac{\hbar\omega}{2} \frac{\sin(\tau(n+1/2))}{\sin(\tau/2)}. \tag{13}$$

In this case one can see [3] that the WKB equivalent potential is given by

$$V(x) = \left(\frac{\tau}{2\sin(\tau/2)}\right)^2 \frac{m\omega^2}{2} x^2$$

$$\left[1 - \frac{8}{15}\left(\frac{x}{2R_e}\right)^4 + \frac{4448}{1575}\left(\frac{x}{2R_e}\right)^8 - \frac{345344}{675675}\left(\frac{x}{2R_e}\right)^{12} + \dots\right], \tag{14}$$

with

$$R_e = \frac{1}{\tau}\left(\frac{\hbar^2}{2m}\right)^{1/2} \left(\frac{2\sin(\tau/2)}{\hbar\omega}\right)^{1/2}. \tag{15}$$

For $\tau = 0$ the usual harmonic oscillator potential is obtained. For $x \to \infty$ the potential goes to a finite limiting value.

For $q = e^\tau$ with τ real, the q-oscillator has a spectrum given by

$$E_n = \frac{\hbar\omega}{2} \frac{\sinh(\tau(n+1/2))}{\sinh(\tau/2)}, \tag{16}$$

while the WKB equivalent potential is given by [3]

$$V(x) = \left(\frac{\tau}{2\sinh(\tau/2)}\right)^2 \frac{m\omega^2}{2} x^2$$

$$\left[1+\frac{8}{15}\left(\frac{x}{2R_h}\right)^4+\frac{4448}{1575}\left(\frac{x}{2R_h}\right)^8+\frac{345344}{675675}\left(\frac{x}{2R_h}\right)^{12}+\ldots\right], \tag{17}$$

with

$$R_h=\frac{1}{\tau}\left(\frac{\hbar^2}{2m}\right)^{1/2}\left(\frac{2\sinh(\tau/2)}{\hbar\omega}\right)^{1/2}. \tag{18}$$

This WKB-EP gives the classical harmonic oscillator potential for $\tau=0$, while it goes to infinity for $x\rightarrow\infty$.

Using the same inversion technique one finds that the WKB equivalent potential for the Q-deformed harmonic oscillator takes the form [7]

$$V(x)=V_{min}+\frac{(\ln Q)^2}{Q}\left(\frac{Q+1}{Q-1}\right)^2\frac{1}{2}m\omega^2x^2$$

$$\left[1-\frac{2}{3}\left(\frac{x}{R'}\right)^2+\frac{23}{45}\left(\frac{x}{R'}\right)^4-\frac{134}{315}\left(\frac{x}{R'}\right)^6+\frac{5297}{14172}\left(\frac{x}{R'}\right)^8-\ldots\right], \tag{19}$$

where

$$V_{min}=\frac{\hbar\omega(\sqrt{Q}-1)}{2\sqrt{Q}(\sqrt{Q}+1)}, \tag{20}$$

and

$$R'=\left(\frac{\hbar\sqrt{Q}(Q-1)}{\omega m(Q+1)}\right)^{1/2}\frac{(\ln Q)^{-1}}{\sqrt{2}}. \tag{21}$$

We now turn our attention to the anharmonic oscillator with $\mathrm{SU}_q(1,1)$ symmetry, which has been found useful in the description of vibrational molecular spectra [2]. The energy spectrum of this oscillator, in the case of complex q, is given by [2]

$$E_n=E_0'-A\frac{\sin(\tau(n-N/2))\sin(\tau(n+1-N/2))}{\sin^2(\tau)}, \tag{22}$$

where

$$N=2n_{max} \qquad \text{or} \qquad N=2n_{max}+1, \tag{23}$$

with n_{max} corresponding to the last level before the dissociation limit. For $\tau\rightarrow 0$ a Morse or modified Pöschl-Teller spectrum is obtained. The WKB equivalent potential in this case is

$$V(x)=V_{min}+\frac{A}{4}\left(\frac{\tau\sin(N\tau)}{\sin^2\tau}\right)^2u^2$$

$$\left[1-\frac{2}{3}\frac{\tau^2\cos(N\tau)}{\sin^2\tau}u^2+\frac{1}{45}\frac{\tau^4(23\cos^2(N\tau)-6)}{\sin^4\tau}u^4\right.$$
$$\left.-\frac{2}{315}\frac{\tau^6(67\cos^2(N\tau)-36)\cos(N\tau)}{\sin^6\tau}u^6+\cdots\right], \tag{24}$$

where

$$V_{min} = E_0' - A\frac{\cos(\tau) - \cos(\tau N)}{2\sin^2\tau}, \tag{25}$$

and

$$u = \frac{\sqrt{2mA}x}{\hbar}. \tag{26}$$

It can be easily seen that for $\tau \to 0$ Eq. (24) reduces to the Taylor expansion of the modified Pöschl-Teller potential

$$V(x) = V_{min} + \frac{A}{4}u^2\left[1 - \frac{2}{3}u^2 + \frac{17}{45}u^4 - \frac{62}{315}u^6 + \dots\right], \tag{27}$$

where

$$u = \frac{\sqrt{2mA}}{\hbar}x, \tag{28}$$

while the modified Pöschl–Teller potential in closed form is

$$V(x) = V_{min} + \frac{AN^2}{4}\tanh^2\left(\frac{\sqrt{2mA}x}{\hbar}\right). \tag{29}$$

This means that the WKB-EP of a system with $SU_q(1,1)$ symmetry is a deformation of the modified Pöschl–Teller potential.

4 Relation to Quasi Exactly Soluble Potentials

On the other hand, it is known [4, 5] that the potentials

$$V(x) = 8a^2x^6 + 8abx^4 + 2[b^2 - (2k+3)a]x^2, \tag{30}$$

with $k = 2n + r$, where $n = 0, 1, 2, \dots$ and $r = 0, 1$ are quasi-exactly soluble. The meaning of this term is the following: for these potentials one can construct exactly the first $n+1$ levels with parity $(-1)^r$. (The extra factors of 2 in eq. (30) in comparison to refs [4, 5] are due to the fact that we use the usual form of the Schrödinger operator

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x), \tag{31}$$

while in refs [4, 5] the form

$$H = -\frac{\hbar^2}{m}\frac{d^2}{dx^2} + V(x) \tag{32}$$

is used.)

We wish to check if the WKB-EPs for the deformed oscillators mentioned above correspond, up to terms of x^6, to quasi-exactly soluble potentials of the form of eq. (30). For reasons of convenience we start the comparison from the $\mathrm{SU}_q(1,1)$ potential of eq. (24), setting $\hbar = m = 1$. Comparing the coefficients of x^2, x^4 and x^6 in eqs (24) and (30) we get respectively the following equations

$$2(b^2 - (2k+3)a) = \frac{A^2}{2} \frac{\tau^2 \sin^2(N\tau)}{\sin^4 \tau}, \tag{33}$$

$$8ab = -\frac{2}{3} A^3 \frac{\tau^4 \sin^2(N\tau) \cos(N\tau)}{\sin^6 \tau}, \tag{34}$$

$$8a^2 = \frac{2}{45} A^4 \frac{\tau^6 \sin^2(N\tau)(23\cos^2(N\tau) - 6)}{\sin^8 \tau}. \tag{35}$$

We can consider $a^2 = 1$ without loss of generality. For the ground state to be normalizable one should have $a \geq 0$ [6], which implies in the present case $a = 1$. Having in mind that in eq. (22) one must have $A > 0$ in order to get an increasing spectrum (this fact can be seen also in the realistic applications to molecular spectra given in ref. [2]), from eq. (35) one finds

$$A = \sqrt{6\sqrt{5}} \frac{\sin^2 \tau}{\tau^{3/2} \sqrt{\sin(N\tau)} \sqrt[4]{23\cos^2(N\tau) - 6}}. \tag{36}$$

Because of the symmetry $q \leftrightarrow q^{-1}$ characterizing the q-deformed oscillators, it suffices to consider $\tau > 0$. The following conditions should then be satisfied

$$\sin(N\tau) > 0, \tag{37}$$

$$\cos^2(N\tau) > 6/23. \tag{38}$$

Then eq. (34) gives

$$b = -\sqrt{\frac{15\sqrt{5}}{2}} \frac{\sqrt{\sin(N\tau)} \cos(N\tau)}{\sqrt{\tau}(23\cos^2(N\tau) - 6)^{3/4}}, \tag{39}$$

while eq. (33) gives

$$2k + 3 = -\frac{3\sqrt{5}}{2} \frac{\sin(N\tau)(18\cos^2(N\tau) - 6)}{\tau(23\cos^2(N\tau) - 6)^{3/2}}. \tag{40}$$

The fact that $k = 2n + r$ has to be positive gives the additional condition

$$\cos^2(N\tau) < 1/3. \tag{41}$$

From eq. (39) it is clear that b can be either positive (when $\cos(N\tau) < 0$) or negative (when $\cos(N\tau) > 0$).

For reasons of convenience, let us consider the case $b > 0$ first. In this case the conditions $\sin(N\tau) > 0$ and $\cos(N\tau) < 0$ imply that we can limit ourselves to the region $\pi > N\tau > \pi/2$. The conditions (38) and (41) then imply that $2.1863 + 2\pi l > N\tau > 2.1069 + 2\pi l$, with $l = 0, 1, 2, \dots$. Thus though we can find an infinity of values for N, as we shall see it suffices to consider $l = 0$.

We first wish to check the accuracy of our approach, which already involves two major approximations: the WKB approximation and the omission in the WKB-EP of terms higher than x^6. In order to achieve this, we will compare the results given by this method to the exact results given in ref. [4] for the case $n = 1$, $r = 0$, in which $2k + 3 = 7$. From eq. (40) one then sees that a possible solution is $N = 151$, $\tau = 0.0144503$, which gives $A = 0.4343473$, $b = 12.589097$, while from eq. (25) (for $V_{min} = 0$) one has $E'_0 = 1636.8943$. We see therefore that eqs (33)–(35) are indeed satisfied and the potentials of eqs (24) and (30) (up to the x^6 term) are identical, being

$$V(x) = 303.02(x^2 + 0.3324x^4 + 0.02640x^6). \tag{42}$$

The two solutions which can be obtained exactly are [4]

$$E_0 = 3b - 2\sqrt{b^2+2} = 12.4307, \qquad E_2 = 3b + 2\sqrt{b^2+2} = 63.1039,$$

$$E_2 - E_0 = 50.6731, \tag{43}$$

while eq. (22) gives the complete spectrum, including

$$E_0 = 12.3805, \qquad E_2 = 63.0584, \qquad E_2 - E_0 = 50.6779. \tag{44}$$

We remark that the agreement between the exact results of ref. [4] and the predictions of eq. (22) is excellent, implying that in this case:

i) the WKB method is accurate,

ii) the omission of terms higher than x^6 is a good approximation.

An additional check for the case $n = 3$, $r = 0$ is also made. In this case $2k + 3 = 15$. One possible solution of eqs (33)-(35) in this case is given by $N = 325$, $\tau = 0.00671384$, $A = 0.2960795$, $b = 18.469158$, $E'_0 = 5168.941$. The potential is

$$V(x) = 652.20(x^2 + 0.2265x^4 + 0.01227x^6). \tag{45}$$

Eq. (22) gives the levels

$$E_0 = 18.1126, \qquad E_2 = 91.3482, \qquad E_4 = 165.8771, \qquad E_6 = 241.6456, \tag{46}$$

while, following the procedure of ref. [4], one sees that in this case the exact

Table 1
Exact energy levels of the quasi-exactly soluble potential of eq. (30) with $n = 9$, $r = 0$, $a = 1$, $b = 21.275801$, compared to the approximate energy levels of the WKB-EP potential of eq. (24) with $A = 0.2703882$, $E'_0 = 7126.0336$, $\tau = 0.00545864$, $N = 399$, given by eq. (22).

n	E_n (approx)	E_n (exact)
0	20.3991	20.4153
2	102.672	102.682
4	186.131	186.138
6	270.735	270.749
8	356.444	356.485
10	443.217	443.317
12	531.013	531.218
14	619.791	620.165
16	709.507	710.133
18	800.118	801.101

energy levels are the roots of the equation

$$E^4-28bE^3+(254b^2-240)E^2+(-812b^3+2592b)E+(585b^4-4656b^2+2880) = 0, \tag{47}$$

given by

$$E_0 = 18.1429, \qquad E_2 = 91.3783, \qquad E_4 = 165.913, \qquad E_6 = 241.703. \tag{48}$$

We remark that excellent agreement between the approximate and the exact results is again obtained.

The agreement remains equally good if more levels are considered. For the case $n = 9$, $r = 0$, in which $2k + 3 = 39$, one possible solution of eqs (33)-(35) is given by $N = 399$, $\tau = 0.00545864$, $A = 0.2703882$, $b = 21.275801$, $E'_0 = 7126.0336$. The potential is

$$V(x) = 827.43(x^2 + 0.2057x^4 + 0.009669x^6).$$

The energy levels are compared to the exact ones in Table I.

We turn now to the case with $b < 0$. The conditions $\sin(N\tau) > 0$ and $\cos(N\tau) > 0$ imply that we should limit ourselves to the region $\pi/2 > N\tau > 0$. Then the conditions (38) and (41) imply that $1.0347 > N\tau > 0.9553$. (We again ignore terms of $2\pi l$.)

We again consider the case $n = 1$, $r = 0$, $2k + 3 = 7$. One possible solution is given by $N = 61$, $\tau = 0.0157377$, $A = 0.4538508$, $b = -12.108743$, $E_0' = 390.66689$. The relevant potential is

$$V(x) = 279.095(x^2 - 0.3471x^4 + 0.02866x^6). \tag{49}$$

The exact solutions given by ref. [4] are

$$E_0 = -60.7083, \qquad E_2 = -11.9441, \tag{50}$$

while eq. (22) gives

$$E_0 = 11.7456, \qquad E_2 = 57.3764. \tag{51}$$

The reasons for this dramatic failure are well understood. The potential of eq. (49) has a central well in the middle, having its minimum at $x = 0$, $E = 0$, plus two symmetrically located wells, on either side of the central one, with their minima at $E < 0$. The method of ref. [4] gives the lowest two energy eigenvalues of the system, which in this case are the lowest two levels in the side wells. The WKB method used in ref. [3], however, is known to be valid only for small values of x, i.e. in the area of the central well. Therefore in this case eq. (51) gives the lowest two levels of the central well, which in this case are *not* the lowest two levels of the system.

We therefore conclude that a correspondence between the $\mathrm{SU}_q(1,1)$ WKB-EP and the quasi-exactly soluble potentials of eq. (30) can be made only for $b > 0$, in which case both methods will give the levels corresponding to a well around $x = 0$. In this case, as we have already seen, the approximation is very good, providing us with the following method:

Given a quasi-exactly soluble potential for which only the $n + 1$ lowest levels of parity r can be found exactly, we can choose appropriately the parameters of an $\mathrm{SU}_q(1,1)$ WKB-EP, so that the two potentials are identical up to order x^6. Then using eq. (22), i.e. the known eigenvalues of the WKB-EP, we can approximate the levels of the same parity lying above the first $n + 1$ known levels, as well as the levels of opposite parity, for which the method of ref. [4] gives no information *for the potential under discussion.* (It is worth noting that changing the values of n and r changes the potential.)

4.2 The q-deformed oscillator

It is now worth examining if the WKB-EPs corresponding to the q-oscillator and the Q-oscillator can be made to correspond to a quasi-exactly soluble potential.

In the case of the q-deformed harmonic oscillator with $q = e^{i\tau}$ (τ real) one should set $b = 0$, since no x^4 terms appear in the WKB-EP of eq. (14). Comparison of the coefficients of x^6 gives then the condition

$$8a^2 = -\frac{\tau^6\omega^4}{240\sin^4(\tau/2)}, \tag{52}$$

which cannot be satisfied. Therefore no connection between this oscillator and the quasi-exactly soluble potentials of eq. (30) is possible.

In the case of the q-deformed harmonic oscillator with $q = e^{\tau}$ (τ real) one should again set $b = 0$, since no x^4 terms appear in the WKB-EP of eq. (17). In addition, because of the symmetry $q \leftrightarrow q^{-1}$ characterizing this oscillator, it suffices to consider $\tau > 0$. Then, taking into account that $a > 0$ for the ground state to be normalizable [6], the coefficients of x^6 give the equation

$$a = \frac{\tau^3\omega^2}{8\sqrt{30}\sinh^2(\tau/2)}, \tag{53}$$

while the coefficients of x^2 give the condition

$$2k + 3 = -\sqrt{\frac{15}{2}}\frac{1}{\tau}. \tag{54}$$

Since both $2k+3$ and τ are positive, this condition cannot be satisfied. Therefore no connection between this oscillator and the quasi-exactly soluble potentials of eq. (30) is possible, either.

4.3 The Q-deformed oscillator

In the case of the Q-deformed oscillator the x^4 term is present in the WKB-EP (see eq. (19)), in contrast to the q-deformed oscillator. In this case the coefficients of the x^6, x^4 and x^2 terms give the equations

$$a = \pm\frac{1}{2}\sqrt{\frac{23}{45}\frac{(\ln Q)^3}{Q}}\left(\frac{Q+1}{Q-1}\right)^2\omega^2, \tag{55}$$

$$b = \mp\frac{1}{2}\sqrt{\frac{5}{23}\frac{\ln Q}{\sqrt{Q}}}\frac{Q+1}{Q-1}\omega, \tag{56}$$

$$2k + 3 = \mp\frac{27}{23}\sqrt{\frac{5}{23}\frac{1}{\ln Q}}. \tag{57}$$

We know that for the ground state to be normalizable one should have $a > 0$. Therefore in eq. (55) the selection of the upper (lower) sign requires $\ln Q > 0$ ($\ln Q < 0$). In both cases it is impossible to satisfy eq. (57). Therefore no connection between the Q-deformed oscillator and the quasi-exactly soluble potentials of eq. (30) is possible.

4.4 The modified Pöschl–Teller potential

In the case of the modified Pöschl-Teller potential, comparing its Taylor expansion (eq. (27)) to the quasi-exactly soluble potentials of eq. (30) and taking into account that $a > 0$ one obtains from the coefficients of the x^6, x^4 and x^2 terms the following equations

$$a = \sqrt{\frac{17}{5}\frac{A^2}{6}}, \tag{58}$$

$$b = -\sqrt{\frac{5}{17}\frac{A}{2}}, \tag{59}$$

$$2k + 3 = -\frac{18}{17}\sqrt{\frac{5}{17}}. \tag{60}$$

Since $2k + 3$ has to be positive, it is clear from the last equation that no connection between the modified Pöschel-Teller potential and the quasi-exactly soluble potentials of eq. (30) is possible.

5 Discussion

The following comments on the results can now be made:

i) The quasi-exactly soluble potentials are known to be related to the SL(2) [5] and deformed SU(2) [8] symmetries. Among the several oscillators considered here, the only one for which the connection to QESPs was possible is the oscillator with $\mathrm{SU}_q(1,1)$ symmetry. Apparently the $\mathrm{H}_q(4)$ symmetry of the q-deformed oscillator is "not enough" to guarantee such a connection.

ii) The QESPs of eq. (30) are characterized by 3 parameters (a, b, k). The $\mathrm{SU}_q(1,1)$ potential is also characterized by 3 parameters (A, τ, N), while the WKB-EPs of the q-oscillator and Q-oscillator are characterized by 2 parameters (ω, τ and ω, Q respectively) and the modified Pöschl-Teller potential is characterized by only one parameter (A). It is therefore not surprising that a connection is possible only for the case in which the number of parameters of the two potentials to be related is the same.

In summary, we have proved that the quasi-exactly soluble potentials corresponding to a harmonic oscillator with quartic and sextic anharmonicities can be approximated by the WKB equivalent potentials corresponding to a deformed anharmonic oscillator with SU$_q$(1,1) symmetry. As a result one can use the relevant SU$_q$(1,1) oscillator in order to approximate the levels of the QESP which cannot be obtained using that approach. In the specific case under discussion the extra levels obtained in this way are the levels of the same parity lying above the first $n+1$ known levels, as well as all the levels of the opposite parity. The fact that the modified Pöschl-Teller potential with SU(1,1) symmetry cannot be connected to these QESPs, while its deformed version with SU$_q$(1,1) symmetry can, is an example where the use of q-deformation allows the solution of an otherwise intractable problem.

In ref. [2] the deformed anharmonic oscillator with SU$_q$(1,1) symmetry has been proved appropriate for the description of vibrational spectra of diatomic molecules. An extension to vibrational spectra of highly symmetric polyatomic molecules has been given in [9]. The construction of QESPs related to realistic vibrational spectra is receiving attention.

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