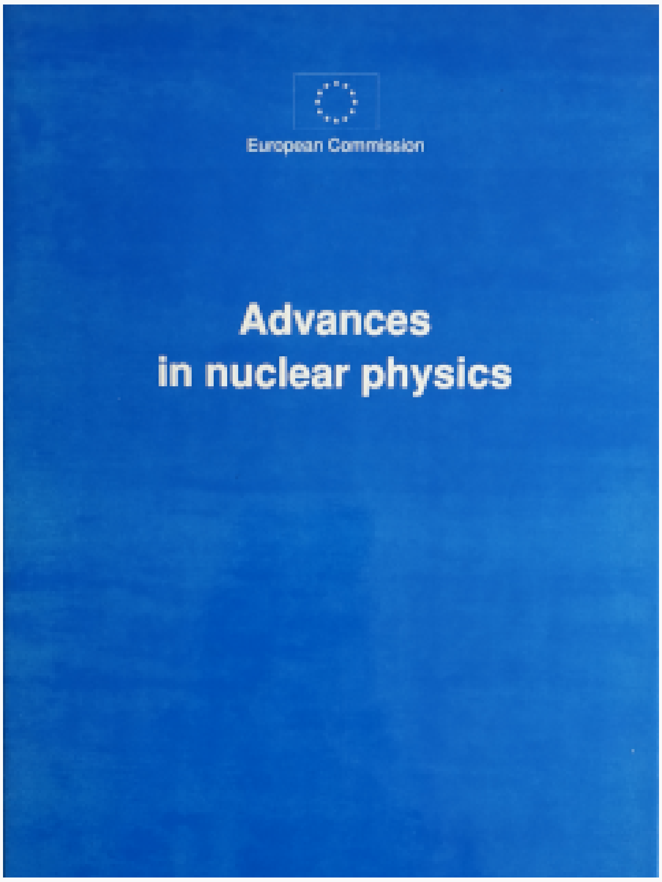
HNPS Advances in Nuclear Physics

Vol 5 (1994)

HNPS1994



A Study of the Electrodynamic Properties of Macroscopically Extended Fermion Matter

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doi: [10.12681/hnps.2905](https://doi.org/10.12681/hnps.2905)

A STUDY OF THE ELECTRODYNAMIC PROPERTIES OF MACROSCOPICALLY EXTENDED FERMION MATTER

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Abstruct:Finitely extended fermion matter is studied in the framework of Dirac's theory.Some asnalytic properties of the spinors are discussed. Propositions are demonstrated important for the formulation of the solution.

I INTRODUCTION

The study of the fermion matter[1,2]is encountering an increasing interest for Astrophysics and particular for the Physics of the neutron stars[3] The neutron stars present real physical objects in which the notion of extended fermion matter finds application.

We are developing a systematic method will allow to study-using the Dira c equation-the finitely extended fermion matter in atomic,nuclear or sub-nuclear forms. The finitely extended fermion systems we consider . fulfill appropriate boundary conditions.

Since nucleons are fermions,it is necessary,to study nuclear matter using the Dirac equation[4] instead the Schroediger equation. This is necessary not only because of the relativistic energies involved,but rather because of the exact description of the spin properties which play the principale role.

More generally,by this method the problem of the quark-star[5-8] matter in the form of strange-quark matter or charm-quark[9] matter can be treated adequately.

In this study we also assume that,nuclear matter interacts besides the nuclear forces also with electromagnetic four-potentials.

The purpose of the present paper is limited in the study of the mathematical properties of spinors[10]that describe nuclear matter under the above conditions. Specifically, we consider the four-potential $(A_0, \mathbf{A})$, where $\mathbf{A}=(A_1, A_2, A_3)$ with $A_k, k = 0,1,2,3$ periodic function in Minkowski space. We take for simplicity A_k as independent of time.

The solution of the Dirac equation, coming after separation of the time variable from the space variables corresponds obviously to the solution of the time independent Dirac equation.

II FORMULATION OF THE PROBLEM

Considered is the equation

$$\left(\sum_{\mu=1}^{3} \delta_\mu(\partial_\mu - i\alpha^\mu) + \alpha^0 + \kappa\gamma^0\right)\Psi = 0, \qquad 2.1$$

in which the following notation is used:

$\alpha^\mu = \frac{eA^\mu}{\hbar c}$; $\alpha^0 = \frac{A^0 - E}{\hbar c} = \frac{mc}{\hbar}$; $\delta_\mu = \gamma^0\gamma^\mu$; $\partial_\mu = \partial/\partial x_\mu$, and γ^μ are the Dirac matrices, where $\mu = 1,2,3$. E is parameter allowing for the time separation and is related to electron energy.

We assume, the A^μ-functions have a period of 2π with respect to the variables x_1, x_2, x_3.

In what follows, we present the most basic results, that are directly related to the solution of 2.1 .

In order to obtain and study this solution, the following definition and propositions are useful:

Definition I

We define the $\mathbb{C}$ linear transformation σ, that operates on the periodic functions f , in the variables x_1, x_2, x_3 of period 2π, in the following way:

For every $\mathbf{n}:=(n_1, n_2, n_3) \in \mathbb{Z}^3$ it is
$(\sigma f)(\mathbf{x}) = f_{\mathbf{n}}$ if $\mathbf{x}:=(x_1, x_2, x_3) \in [n_1, n_1+1) \times [n_2, n_2+1) \times [n_3, n_3+1) \subset \mathbb{R}^3$ were $f_{\mathbf{n}}$ are the expansion coefficients of the function f in Fourier series.

In other words σf is a step function defined in $\mathbb{R}^3$. If the periodic functions, $a^{\mu}, \mu=0,1,2,3$, are continuous in $\mathbb{R}^3$ and satisfy the condition:

$$\sum_{n\in\mathbb{Z}^3} |a_n^{\mu}| < \infty$$

where a_n^{μ} , $n \in \mathbb{Z}$ are the coefficients of the expansion of a^{μ} in Fourier series, then there holds the following:

Proposition I

Given Spinor $\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{bmatrix}$ defined in $\mathbb{R}^3$, such that:

Ψ_v, $\partial_{\mu}\Psi_v$, $x_1x_2x_3\Psi_v \in L^2(\mathbb{R}^3)$ and $\partial_{\mu}\Psi_v$ are continous functions in $\mathbb{R}^3$ and ψ_v bounded functions in $\mathbb{R}^3$ for $v=1,2,3,4$ and $\mu=1,2,3$. Then the following statement is true:

The spinor Ψ is a solution of (1) iff $\hat{\Psi}$ is a solution of the equation:

$$\int_{\mathbb{R}^3} [\ -i \sum_{\mu=1}^{3} \delta_{\mu}(\sigma a^{\mu})(s) + (\sigma a^0)(s)\ ]\ (\partial_1\partial_2\partial_3\hat{\Psi})(k+s)d^3s$$

$$= \int_{k_1}^{k_1+1}\int_{k_2}^{k_2+1}\int_{k_3}^{k_3+1} \partial_1\partial_2\partial_3[i \sum_{\mu=1}^{3} k'_{\mu}\delta_{\mu} - \gamma^0\ ]\hat{\Psi}(k')\ dk'_1dk'_2dk'_3\ , \qquad 2.2$$

where $\hat{\Psi}$ is the Fourier transform of Ψ.

With the aid of **Proposition I**, we prove the following basic

Theorem I

Let spinor ψ be a solution of equation (1), Ψ satisfy the pressupositions of the above proposition and the following conditions:

$$\partial_1\partial_2\partial_3(k_{\mu}\hat{\Psi}_v(k)) \in L^1(\mathbb{R}^3) \qquad 2.3$$

$$\lim_{k_1\to\pm\infty}\ \lim_{k_2\to\pm\infty}\ \lim_{k_3\to\pm\infty}\ (k_{\mu}\hat{\Psi}_v(k)) = 0 \qquad 2.4$$

for $\mu=1,2,3$; $v=1,2,3,4$.

Then, a) spinor Ψ satisfies

$$\Big(\sum_{\mathbf{n}\in\mathbb{Z}^3} \exp(i\mathbf{n}\mathbf{x})\Big)\Big(\prod_{\nu=1}^{3}[\exp(ix_\nu)-1]\Big)\Big(\{\sum_{\mu=1}^{3}\delta_\mu a^\mu + ia^0\}\Psi\Big) = 0 \qquad 2.5$$

β) If in addition, spinor $\hat{\Psi}$ is continuous in $\mathbb{R}^3$, then $\Psi = 0$. For the proof of **Proposition I** and **Theorem I**, the following notation and remarks are required:

Notation and Remarks:

α) $\partial_\mu := \frac{\partial}{\partial x_\mu}$; $(\partial_\mu u)(\mathbf{m}) := \partial_\mu u(x)\,|_{\mathbf{x}=\mathbf{m}}$.

β) $\mathbb{L}^p := \mathbb{L}^p(\mathbb{R}^3)$ is the set of funktions u defined in $\mathbb{R}^3$ for wich there holds:

$$\int_{\mathbb{R}^3} |u(\mathbf{x})|^p\, d^3\mathbf{x} < \infty, \text{ where } \mathbf{x}=(x_1,x_2,x_3),\ p \in \mathbb{R}^+.$$

γ) $\mathbb{F}_{(2\pi,3)}$ is the set of functions defined in $\mathbb{R}^3$, and are periodic with respect to the variables x_1, x_2, x_3 with the same period 2π and are integrable on $[0,2\pi] \times [0,2\pi] \times [0,2\pi] \subset \mathbb{R}^3$.

It is well known that:

If $f \in \mathbb{F}_{(2\pi,3)}$ then $f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} f_{\mathbf{n}} \exp(i\mathbf{n}\mathbf{x})$,

where

$$\mathbf{n}\mathbf{x} := n_1x_1 + n_2x_2 + n_3x_3 \text{ with } \mathbf{n}=(n_1,n_2,n_3)\ ,\mathbf{x}=(x_1,x_2,x_3)$$

$$f_{\mathbf{n}} = (2\pi)^{-3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(\mathbf{x})\exp(i\mathbf{n}\mathbf{x})\,d^3\mathbf{x}\ .$$

δ) $\mathbb{F}^1_{(2\pi,3)} := \{ f \in \mathbb{F}_{(2\pi,3)}: \sum_{\mathbf{n}\in\mathbb{Z}^3} |f_{\mathbf{n}}| < \infty \}$.

Consequently, the functions a^μ of equation (1) are elements of $\mathbb{F}^1_{(2\pi,3)}$.

ε) If $\mathbf{n} := (n_1,n_2,n_3)$ and $\mathbf{1} := (1,1,1)$ we define: $[\mathbf{n},\mathbf{n}+\mathbf{1}) := [n_1,n_1+1) \times [n_2,n_2+1) \times [n_3,n_3+1)$ (semi-open cube)

Obviously, for $\mathbf{k} \in \mathbb{R}^3$, the set of $[\mathbf{n}+\mathbf{k},\mathbf{n}+\mathbf{k}+\mathbf{1}), \mathbf{n} \in \mathbb{Z}^3$ is a partition of $\mathbb{R}^3$. That is $\bigcup_{\mathbf{n}\in\mathbb{Z}^3}[\mathbf{n}+\mathbf{k},\mathbf{n}+\mathbf{k}+\mathbf{1}) = \mathbb{R}^3$ and $[\mathbf{n}+\mathbf{k},\mathbf{n}+\mathbf{k}+\mathbf{1}) \cap [\mathbf{m}+\mathbf{k},\mathbf{m}+\mathbf{k}+\mathbf{1}) = \emptyset$ for $\mathbf{n}\neq\mathbf{m}$. Thus, we have the

Remark 1

For every integrable funktion u on $\mathbb{R}^3$ and for every $\mathbf{k} \in \mathbb{R}^3$ is

$$\sum_{\mathbf{n}\in\mathbb{Z}^3} \int_{\mathbf{n}}^{\mathbf{n+1}} u(\mathbf{x})d^3\mathbf{x} = \int_{\mathbb{R}^3} u(\mathbf{x})d^3\mathbf{x} = \sum_{\mathbf{n}\in\mathbb{Z}^3} \int_{\mathbf{n+k}}^{\mathbf{n+k+1}} u(\mathbf{x})d^3\mathbf{x},$$

where $\displaystyle\int_{\mathbf{a}}^{\mathbf{a+1}} u(\mathbf{x})d^3\mathbf{x} := \int_{a_1}^{a_1+1}\int_{a_2}^{a_2+1}\int_{a_3}^{a_3+1} u(x_1,x_2,x_3)\, dx_3dx_2dx_1$

στ) We define the $\mathbb{C}$-linear transformation T in the set of functions $u \in \mathbb{R}^3$,in the fllowing way:

$$(Tu)(\mathbf{x}):=u(x_1+1,x_2+1,x_3+1)-u(x_1+1,x_2+1,x_3)-u(x_1+1,x_2,x_3+1)$$
$$-u(x_1,x_2+1,x_3+1)+u(x_1+1,x_2,x_3)+u(x_1,x_2+1,x_3)+u(x_1,x_2,x_3+1)$$
$$-u(x_1,x_2,x_3).$$

Remark 2

If the function $\partial_1\partial_2\partial_3 u$ is defined for every $\mathbf{x} \in \mathbb{R}^3$ and is integrable on $\mathbb{R}^3$,then it holds:

$$(Tu)(\mathbf{x}) = \int_{\mathbf{x}}^{\mathbf{x+1}} \partial_1\partial_2\partial_3\, u(s)d^3s,$$

$$(Tu)(\mathbf{k+n}) = \int_{\mathbf{n}}^{\mathbf{n+1}} (\partial_1\partial_2\partial_3 u)(\mathbf{k+s})d^3s,$$

where $\mathbf{x} =(x_1,x_2,x_3)$.

Proof:

$$\int_{\mathbf{n}}^{\mathbf{n+1}} (\partial_1\partial_2\partial_3 u)(\mathbf{k+s})d^3\mathbf{s} = \int_{\mathbf{n+k}}^{\mathbf{k+n+1}} \partial_1\partial_2\partial_3 u(\mathbf{x})d^3\mathbf{x} \qquad (\mathbf{x=k+s})$$

$$= \int_{k_3+n_3}^{k_3+n_3+1}\int_{k_2+n_2}^{k_2+n_2+1} [\partial_2\partial_3 u(k_1+n_1+1,x_2,x_3)-\partial_2\partial_3 u(k_1+n_1,x_2,x_3)]dx_2dx_3$$

$$= \int_{k_3+n_3}^{k_3+n_3+1} [\partial_3 u(k_1+n_1+1,k_2+n_2+1,x_3)-\partial_3 u(k_1+n_1+1,k_2+n_2,x_3)$$

$$-\partial_3 u(k_1+n_1,k_2+n_2+1,x_3)+\partial_3 u(k_1+n_1,k_2+n_2,x_3)]dx_3$$

$$=u(k_1+n_1+1,k_2+n_2+1,k_3+n_3+1)\ -u(k_1+n_1+1,k_2+n_2+1,k_3+n_3)-\ \ldots$$

$$-u(k_1+n_1,k_2+n_2,k_3+n_3)\ =\ (Tu)(\mathbf{k}+\mathbf{n}).$$

The first relation can be proved in a similar way.

ζ) With $\mathbb{M}_1^3$ we will symbolize the set of step-functions t defined in $\mathbb{R}^3$, that satisfy:

ζ_1) For every $\mathbf{x} \in [\mathbf{n},\mathbf{n}+1)$ there holds true: $t(\mathbf{x})=t_{\mathbf{n}}$ ($t_{\mathbf{n}}$ is a complex constant that depends on $\mathbf{n}$), where $\mathbf{n} \in \mathbb{Z}^3$.

ζ_2) $t \in \mathbb{L}^1$

η) I consider the $\mathbb{C}$-linear transformation σ, that operates on the elements f of $\mathbb{F}^1_{(2\pi,3)}$ as follows:

$$(\sigma f)(\mathbf{x})=\ f_{\mathbf{n}}\ \text{av}\ \mathbf{x} \in [\mathbf{n},\mathbf{n}+1)\ ,\mathbf{n} \in \mathbb{Z}^3,$$

where $f(\mathbf{x})=\sum_{\mathbf{n}\in\mathbb{Z}^3} f_{\mathbf{n}}\exp(i\mathbf{n}\mathbf{x})$.

This means, that the elements of $\sigma(\mathbb{F}^1_{(2\pi,3)})$ fufill the condition ζ_1), which satisfy also the elements of $\mathbb{M}_1^3$. Specifically there holds true

Remark 3

From $f \in \mathbb{F}^1_{(2\pi,3)}$ we obtain: $\sigma f \in \mathbb{M}_1^3$.

Proof:

It is sufficient, to prove that $\sigma f \in \mathbb{L}^1$:

From $f \in \mathbb{F}^1_{(2\pi,3)}$, we obtain, that if $\mathbf{x} \in [\mathbf{n},\mathbf{n}+1)$, then $(\sigma f)(\mathbf{x})=f_{\mathbf{n}} \in \mathbb{C}$ and $\sum_{\mathbf{n}\in\mathbb{Z}^3} |f_{\mathbf{n}}| < \infty$. So it holds:

$$\int_{\mathbb{R}^3} |(\sigma f)(\mathbf{x})|d^3\mathbf{x} = \sum_{\mathbf{n}\in\mathbb{Z}^3} \int_{\mathbf{n}}^{\mathbf{n}+1} |(\sigma f)(\mathbf{x})|d^3\mathbf{x} \quad \text{(Due to remark 1)}$$

$$=\sum_{\mathbf{n}\in\mathbb{Z}^3} \int_{\mathbf{n}}^{\mathbf{n}+1} |f_{\mathbf{n}}|d^3\mathbf{x}\ =\ \sum_{\mathbf{n}\in\mathbb{Z}^3} |f_{\mathbf{n}}|\ <\ \infty$$

Remark 4

Let $f \in \mathbb{F}^1_{(2\pi,3)}$ with $f(\mathbf{k}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} f_{\mathbf{n}}\exp(i\mathbf{n}\mathbf{k})$ then the inverse

Fourier transform of $\sigma(f)$ exists and it is:

$$(2\pi)^{-3/2}\int_{\mathbb{R}^3}(\sigma(f))(\mathbf{k})\exp(i\mathbf{kx})d^3\mathbf{k}$$

$$= (2\pi)^{-3/2} i\frac{[\exp(ix_1)-1][\exp(ix_2)-1][\exp(ix_2)-1]}{x_1x_2x_3}f(\mathbf{x}).$$

Proof:

From $f \in \mathbb{F}^1_{(2\pi,3)} \subset \mathbb{L}_1(\mathbb{R}^3)$ we have that there exists the Fourier transform of f.Additionaly,from the definition of σ and with the aid of remark 1,it follows:

$$\int_{\mathbb{R}^3}(\sigma f)(\mathbf{k})\exp(i\mathbf{kx})d^3\mathbf{k} = \sum_{\mathbf{n}\in\mathbb{Z}^3}\int_{\mathbf{n}}^{\mathbf{n}+1}(\sigma f)(\mathbf{k})\exp(i\mathbf{kx})d^3\mathbf{k}$$

$$= \sum_{\mathbf{n}\in\mathbb{Z}^3}\int_{\mathbf{n}}^{\mathbf{n}+1} f_{\mathbf{n}}\exp(i\mathbf{kx})d^3\mathbf{k}$$

$$= \sum_{\mathbf{n}\in\mathbb{Z}^3} f_{\mathbf{n}}\{\int_{n_1}^{n_1+1}\exp(ik_1x_1)dk_1\}\{\int_{n_2}^{n_2+1}\exp(ik_2x_2)dk_2\}\{\int_{n_3}^{n_3+1}\exp(ik_3x_3)dk_3\}$$

$$= \sum_{\mathbf{n}\in\mathbb{Z}^3} f_{\mathbf{n}} \prod_{\mu=1}^{3}\{\frac{1}{x_\mu}[\exp(i(n_\mu+1)x_\mu)-\exp(in_\mu x_\mu)]\}$$

$$= \Big(\prod_{\mu=1}^{3}\{\frac{1}{x_\mu}[\exp(ix_\mu)-1]\}\Big)\sum_{\mathbf{n}\in\mathbb{Z}^3} f_{\mathbf{n}}\exp(i\mathbf{nx}).$$

Remark 5

if $u \in \mathbb{L}^2$ then obviously $Tu \in \mathbb{L}$.

In addition there holds:

$$\int_{\mathbb{R}^3}(Tu)(\mathbf{k})\ \exp(-i\mathbf{kx})d^3\mathbf{k} =\{\prod_{l=1}^{3}(\exp(ix_l)-1)\}\int_{\mathbb{R}^3}u(\mathbf{k})\exp(-i\mathbf{kx})d^3\mathbf{k}.$$

Proof:

From the definition of T we obtain:

$$\int_{\mathbb{R}^3}(Tu)(\mathbf{k})\exp(-i\mathbf{kx})d^3\mathbf{k} = \int_{\mathbb{R}^3} u(k_1+1,k_2+1,k_3+1)\exp(-i\mathbf{kx})d^3\mathbf{k}$$

$$-\int_{\mathbb{R}^3} u(k_1+1,k_2+1,k_3)\exp(-i\mathbf{kx})d^3\mathbf{k}-\int_{\mathbb{R}^3} u(k_1+1,k_2,k_3+1)\exp(-i\mathbf{kx})d^3\mathbf{k}$$

$$-\int_{\mathbb{R}^3} u(k_1,k_2+1,k_3+1)\exp(-i\mathbf{kx})d^3\mathbf{k} + \int_{\mathbb{R}^3} u(k_1+1,k_2,k_3)\exp(-i\mathbf{kx})d^3\mathbf{k}$$

$$+\int_{\mathbb{R}^3} u(k_1,k_2+1,k_3)\exp(-i\mathbf{kx})d^3\mathbf{k} + \int_{\mathbb{R}^3} u(k_1,k_2,k_3+1)\exp(-\mathbf{kx})d^3\mathbf{k}$$

$$- \int_{\mathbb{R}^3} u(k_1,k_2,k_3)\exp(-i\mathbf{kx})d^3\mathbf{k}.$$

Performing the substitution $\mathbf{k}'= \mathbf{k} +\mathbf{1}$ i.e $k_1'=k_1+1,k_2'=k_2+1, k_3'=k_3+1$ in the first integral of the second term,and $k_1'=k_1+1, k_2'=k_2+1,k_3'=k_3$ in the second integral of the second term e.t.c we obtain:

$$\int_{\mathbb{R}^3} (Tu)(\mathbf{k})\exp(-i\mathbf{kx})d^3\mathbf{k}$$

$$=\exp(ix_1)\exp(ix_2)\exp(ix_3)\ I(\mathbf{x}) - \exp(ix_1)\exp(ix_2)\ I(\mathbf{x})$$

$$- \exp(ix_1)\exp(ix_3)\ I(\mathbf{x}) - \exp(ix_2)\exp(ix_3)\ I(\mathbf{x})$$

$$+ \exp(ix_1)\ I(\mathbf{x}) + \exp(ix_2)\ I(\mathbf{x}) + \exp(ix_3)\ I(\mathbf{x}) - I(\mathbf{x})$$

$$= \{(\exp(ix_1)-1)(\exp(ix_2)-1)(\exp(ix_3)-1)\}\ I(\mathbf{x}),$$

where :

$$I(\mathbf{x}):= \int_{\mathbb{R}^3} u(\mathbf{k}')\exp(-i\mathbf{k}'\mathbf{x})d^3\mathbf{k}'.$$

III STUDY OF THE SPINOR PROPERTIES - PROOF OF A PROPOSITION

In section we prove the basic properties of the spinors which may be used in the exact study of nuclear matter properties. The relations to be roved enable us to represent with the help of elementary functions the solution of the Dirac equation for the extended nuclear matter.

Proof of the proposition I:

a) Let Ψ be a solution of 2.1 .From $a^{\mu} \in \mathbb{F}^{1}_{(2\pi,3)}$,it follows:

$$a^{\mu}=\sum_{\mathbf{n}\in\mathbb{Z}^3} a^{\mu}_{\mathbf{n}}\exp(i\mathbf{n}x),\ \mu=0,1,2,3 \qquad 3.1$$

Since $\hat{\Psi}$ is the Fourier-transform of Ψ,we will have:

$$\Psi=(2\pi)^{-3/2}\int_{\mathbb{R}^3}\hat{\Psi}(\mathbf{k})\exp(-i\mathbf{k}x)d^3\mathbf{k} \qquad 3.2$$

Using the relations 3.1-3.2 in 2.1 we obtain:

$$\Big\{ \int_{\mathbb{R}^3} \Big(\sum_{\mu=1}^{3}\Big[-ik_{\mu}\delta_{\mu}-i\sum_{\mathbf{n}\in\mathbb{Z}^3}\delta_{\mu}a^{\mu}_{\mathbf{n}}\exp(i\mathbf{n}x)\Big]$$

$$+\sum_{\mathbf{n}\in\mathbb{Z}^3} a^{0}_{\mathbf{n}}\exp(i\mathbf{n}x)+\kappa\gamma^0\Big)\hat{\Psi}(\mathbf{k})\exp(-i\mathbf{k}x)d^3\mathbf{k}=0\Big\}$$

$$\Rightarrow\Big\{\int_{\mathbb{R}^3}\Big(\sum_{\mu=1}^{3}-ik_{\mu}\delta_{\mu}+\kappa\gamma^0\Big)\hat{\Psi}(\mathbf{k})\exp(-i\mathbf{k}x)d^3\mathbf{k}$$

$$=\int_{\mathbb{R}^3}\Big(\sum_{\mathbf{n}\in\mathbb{Z}^3}\{i\sum_{\mu=1}^{3}\delta_{\mu}a^{\mu}_{\mathbf{n}}-a^{0}_{\mathbf{n}}\}\Big)\hat{\Psi}(\mathbf{k})\exp(i(\mathbf{n}-\mathbf{k})x)d^3\mathbf{k}\Big\}.$$

Interchanging integration and summation on the rhs yieldes the relation (see Appendix)

$$\int_{\mathbb{R}^3}\Big[\sum_{\mu=1}^{3}-ik_{\mu}\delta_{\mu}+\kappa\gamma^0\Big]\hat{\Psi}(\mathbf{k})\exp(-i\mathbf{k}x)d^3\mathbf{k}$$

$$=\sum_{\mathbf{n}\in\mathbb{Z}^3}\Big[i\sum_{\mu=1}^{3}a^{\mu}_{\mathbf{n}}\delta_{\mu}-a^{0}_{\mathbf{n}}\Big]\int_{\mathbb{R}^3}\exp(i(\mathbf{n}-\mathbf{k})x)\hat{\Psi}(\mathbf{k})d_3\mathbf{k}.$$

Furthermore,we substitute $\mathbf{n}-\mathbf{k}$ by $-\mathbf{k}'$in each integral of the second part and we get

$$\int_{\mathbb{R}^3}\Big(\sum_{\mu=1}^{3}-ik_{\mu}\delta_{\mu}+\kappa\gamma^0\Big)\hat{\Psi}(\mathbf{k})\exp(-i\mathbf{k}x)d^3\mathbf{k}$$

$$=\sum_{\mathbf{n}\in\mathbb{Z}^3}\int_{\mathbb{R}^3}\Big(\sum_{\mu=1}^{3}ia^{\mu}_{\mathbf{n}}\delta_{\mu}-a^{0}_{\mathbf{n}}\Big)\hat{\Psi}(\mathbf{n}+\mathbf{k}')\exp(-i\mathbf{k}'x)d^3\mathbf{k}'.$$

By again interchanging summation and integration on the rhs we have

$$\int_{\mathbb{R}^3} \Big(\sum_{\mu=1}^{3} -ik_\mu\delta_\mu + \kappa\gamma^0 \Big)\hat{\Psi}(\mathbf{k})\exp(-i\mathbf{kx})d^3\mathbf{k}$$

$$= \int_{\mathbb{R}^3} \sum_{\mathbf{n}\in\mathbb{Z}^3} \Big(\sum_{\mu=1}^{3} ia^\mu_{\mathbf{n}}\delta_\mu - a^0_{\mathbf{n}}\Big)\hat{\Psi}(\mathbf{n}+\mathbf{k})\exp(-i\mathbf{kx})d^3\mathbf{k} \ .$$

According to the conditions of the proposition is the first term of the last equation a spinor whose components are in $\mathbb{L}^2$.This entails the same property for the rhs of the last equation.

Consequently,we immeadiately from the last equation the relation:

$$\Big(-i\sum_{\mu=1}^{3} k_\mu\delta_\mu + \kappa\gamma^0\Big)\hat{\Psi}(\mathbf{k})$$

$$= \sum_{\mathbf{n}\in\mathbb{Z}^3} \Big(i\sum_{\mu=1}^{3} a^\mu_{\mathbf{n}}\delta_\mu - a^0_{\mathbf{n}} \Big)\hat{\Psi}(\mathbf{n} + \mathbf{k}).$$

Applying transformation T (defined in Remark 2) on the last equation and using afterwards Remark (2) on every term of the second part,we obtain

$$T\Big\{ \Big(-i\sum_{\mu=1}^{3} k_\mu\delta_\mu + \kappa\gamma^0\Big)\hat{\Psi}(\mathbf{k}) \Big\}$$

$$= \sum_{\mathbf{n}\in\mathbb{Z}^3} \int_{\mathbf{n}}^{\mathbf{n}+1} \Big(i\sum_{\mu=1}^{3} a^\mu_{\mathbf{n}}\delta_\mu - a^0_{\mathbf{n}} \Big)(\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{s}+\mathbf{k})d^3\mathbf{s} \ \Rightarrow$$

Introducing σa^μ(see definition of σ) in the second part of the equation we have:

$$T\Big\{\Big(-i\sum_{\mu=1}^{3} k_\mu\delta_\mu + \kappa\gamma^0\Big)\hat{\Psi}(\mathbf{k})\Big\}$$

$$= \sum_{\mathbf{n}\in\mathbb{Z}^3} \int_{\mathbf{n}}^{\mathbf{n}+1} \Big(i\sum_{\mu=1}^{3} (\sigma a^\mu)\delta_\mu - \sigma a^0\Big)(\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{s}+\mathbf{k})\ d^3\mathbf{s}.$$

We apply Remark 2 on the lhs and Remark 1 on the rhs we gwt immediately equation 2.2.

β) Let $\hat{\Psi}$ be a solution of equation 2.2. Carrying out the substitution $\mathbf{s}=-\mathbf{x}$ in the integral of the lhs of 2.2 and using

Remark 2, in the rhs we obtain:

$$\int_{\mathbb{R}^3}[\ -i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(-\mathbf{x})\ +(\sigma a^0)(-\mathbf{x})]\ (\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k}-\mathbf{x})d^3\mathbf{x}$$

$$= T\{[\ i\sum_{\mu=1}^{3}k_\mu\delta_\mu\ -\ \kappa\gamma^0\ ]\hat{\Psi}(\mathbf{k})\}$$

Next, we multiply the equation with exp(-$i\mathbf{k}\mathbf{s}$) and integrate on $\mathbb{R}^3$. The following result is obtained:

$$\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}[-i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(-\mathbf{x})+(\sigma a^0)(-\mathbf{x})](\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k}-\mathbf{x})d^3\mathbf{x}\Big)\exp(-i\mathbf{k}\mathbf{s})d^3\mathbf{k}=$$

$$=\int_{\mathbb{R}^3}T\{[\ i\sum_{\mu=1}^{3}k_\mu\delta_\mu\ -\ \kappa\gamma^0\ ]\hat{\Psi}(\mathbf{k})\}\ \exp(-i\mathbf{k}\mathbf{s})d^3\mathbf{k}. \qquad 3.3$$

Now, we observe that, according to the condition of the proposition $x_1x_2x_3\Psi_v\in\mathbb{L}^2$ and it follows that the components of $\partial_1\partial_2\partial_3\hat{\Psi}$ are elements of $\mathbb{L}^2$.

The same holds true, also, for the components of $-i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(-\mathbf{x})$ $+(\sigma a^0)(-\mathbf{x})$. The last assertion is obtained in the following way:

From $a_\mu\in\mathbb{F}^1_{(2n,3)}$ it follows, according to Remark 2.3, that

$$\sigma a_\mu\in\mathbb{M}^1_3\subset\mathbb{L}^1\subset\mathbb{L}^2$$

It follows, therefore, that the components of $-i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(-\mathbf{x})$

+

$(\sigma a^0)(-\mathbf{x})$ will be elements of $\mathbb{L}^2$, since they are linear combinations of σa^μ, μ=0,1,2,3. Consequently, we can apply the convolution theorem on the first part of equation 3.3:

$$\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}[-i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(-\mathbf{x})+(\sigma a^0)(-\mathbf{x})](\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k}-\mathbf{x})d^3\mathbf{x}\Big)\exp(-i\mathbf{k}\mathbf{s})d^3\mathbf{k}$$

$$=\int_{\mathbb{R}^3}\{-i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(-\mathbf{k})+(\sigma a^0)(-\mathbf{k})\}\exp(-i\mathbf{k}\mathbf{s})d^3\mathbf{k}$$

$$\times\int_{\mathbb{R}^3}(\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k})\exp(-i\mathbf{k}\mathbf{s})d^3\mathbf{k} \qquad 3.4$$

We observe that, if we make the substitution $\mathbf{k}=-\mathbf{t}$ in the integrals of the rhs and subsequently we use Remark 4 we get:

$$\int_{\mathbb{R}^3} \{-i\sum_{\mu=1}^{3} \delta_\mu(\sigma a^\mu)(-\mathbf{k})+(\sigma a^0)(-\mathbf{k})\}\exp(-i\mathbf{k}s)d^3\mathbf{k}$$

$$=\int_{\mathbb{R}^3} \{-i\sum_{\mu=1}^{3} \delta_\mu(\sigma a^\mu)(\mathbf{t})+(\sigma a^0)(\mathbf{t})\}\exp(i\mathbf{t}s)d^3\mathbf{t}$$

$$= \frac{[\exp(is_1)-1][\exp(is_2)-1][\exp(is_3)-1]}{s_1 s_2 s_3}$$

$$\times\Big(\sum_{\mu=1}^{3} \delta_\mu a^\mu(s) + ia^0(s)\Big). \qquad 3.5$$

For the other integral of the rhs of 2.9 there holds true according to the property of Fourier transforms
$\partial_\mu\Psi(s)=\int_{\mathbb{R}^3} -ik_\mu\hat{\Psi}(\mathbf{k})\exp(-i\mathbf{k}s)d^3\mathbf{k}$.

$$\int_{\mathbb{R}^3} (\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k})\exp(-i\mathbf{k}s)d^3\mathbf{k} = -is_1 s_2 s_3\Psi(s). \qquad 3.6$$

On the other hand, using Remark 5 and basic properties of Fourier transforms we obtain from the rhs of 3.3:

$$\int_{\mathbb{R}^3} T\{[\ i\sum_{\mu=1}^{3} k_\mu\delta_\mu - \gamma^0]\hat{\Psi}(\mathbf{k})\}\ \exp(-i\mathbf{k}s)d^3\mathbf{k}$$

$$=\{\prod_{\nu=1}^{3} [\exp(is_\nu)-1]\}\int_{\mathbb{R}^3} \{i\sum_{\mu=1}^{3} k_\mu\delta_\mu - \gamma^0\}\hat{\Psi}(\mathbf{k})\exp(-i\mathbf{k}s)d^3\mathbf{k}$$

$$= \{\prod_{\nu=1}^{3} [\exp(is_\nu)-1]\}\Big((-\sum_{\mu=1}^{3} \delta_\mu\partial_\mu - \gamma^0)\Psi(s)\Big) \qquad 3.7$$

Due to the relations 3.4-3.7 equation 3.3 becomes:

$$\{\prod_{l=1}^{3} [\exp(is_l)-1]\}\Big(\sum_{\mu=1}^{3} \delta_\mu(\partial_\mu - ia^\mu) + \kappa\gamma^0\Big)\Psi(s) = 0 \qquad 3.8$$

Let there hold

$$f(s):=\{\prod_{l=1}^{3} [\exp(is_l)-1]\}$$

$$\theta(s) := \left(\sum_{\mu=1}^{3} \delta_\mu(\partial_\mu - ia^\mu) + \kappa\gamma^0 \right)\Psi(s)$$

$$\mathbb{E}_1 := \left\{(2n\pi, s_2, s_3) \; : \; n \in \mathbb{Z} \;\; ; \; s_2, s_3 \in \mathbb{R}\right\},$$

$$\mathbb{E}_2 := \left\{(s_1, 2n\pi, s_3) \; : \; n \in \mathbb{Z} \;\; ; \; s_1, s_3 \in \mathbb{R}\right\},$$

$$\mathbb{E}_3 := \left\{(s_1, s_2, 2n\pi) \; : \; n \in \mathbb{Z} \;\; ; \; s_1, s_2 \in \mathbb{R}\right\},$$

$$\mathbb{E} := \mathbb{E}_1 \cup \mathbb{E}_2 \cup \mathbb{E}_3 .$$

Obviously the set of solutions of equation $f(s) = 0$ in $\mathbb{R}^3$ is identical with the set $\mathbb{E}$. Consequently,from equation 3.8 e obtain for every $s \in \mathbb{R}^3 - \mathbb{E}$ valid the equation:$\theta(s) = 0$.
Since,now,obviously $\mathbb{R}^3 - \mathbb{E}$ is dense in $\mathbb{R}^3$ (with respect to the physical topology of $\mathbb{R}^3$) and θ is continuous in $\mathbb{R}^3$ (according to the conditions of the proposition) we have: $\theta(s) = 0$ for every $s \in \mathbb{R}^3$.

IV FURTHER SPINOR PROPERTIES-PROOF OF THE THEOREM I

a) Since Ψ is a solution of equation 2.1 spinor $\hat{\Psi}$ will be solution of 2.2 (according to proposition I).Then:

$$\int_{\mathbb{R}^3} [\; -i \sum_{\mu=1}^{3} \delta_\mu (\sigma a^\mu)(x) + (\sigma a^0)(x)] \; (\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k} + \mathbf{x}) d^3\mathbf{x}$$
$$= \int_{\mathbf{k}}^{\mathbf{k+1}} \partial_1\partial_2\partial_3 \left([i \sum_{\mu=1}^{3} k'_\mu \delta_\mu - \gamma^0]\hat{\Psi}(\mathbf{k}')\right) d^3\mathbf{k}' \qquad 4.1$$

placing in (14) $\mathbf{k+n}$,in place of $\mathbf{k}$ for every $\mathbf{n} \in \mathbb{Z}^3$,we obtain:

$$\int_{\mathbb{R}^3} [\; -i \sum_{\mu=1}^{3} \delta_\mu (\sigma a^\mu)(x) + (\sigma a^0)(x)] \; (\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k+n+x}) d^3\mathbf{x} =$$
$$= \int_{\mathbf{k+n}}^{\mathbf{k+n+1}} \partial_1\partial_2\partial_3 \left([i \sum_{\mu=1}^{3} k'_\mu \delta_\mu - \kappa\gamma^0]\hat{\Psi}(\mathbf{k}')\right) d^3\mathbf{k}' \qquad 4.2$$

Summing the equations 4.2 part by part for every $\mathbf{n} \in \mathbb{Z}^3$,we obtain:

$$\sum_{\mathbf{n}\in\mathbb{Z}^3}\int_{\mathbb{R}^3}[\ -i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(\mathbf{x})\ +(\sigma a^0)(\mathbf{x})]\ (\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k}+\mathbf{n}+\mathbf{x})d^3\mathbf{x} =$$

$$= \sum_{\mathbf{n}\in\mathbb{Z}^3}\int_{\mathbf{k}+\mathbf{n}}^{\mathbf{k}+\mathbf{n}+1}\partial_1\partial_2\partial_3\Big([i\sum_{\mu=1}^{3}k'_\mu\delta_\mu - \kappa\gamma^0]\hat{\Psi}(\mathbf{k}')\Big)d^3\mathbf{k}' \quad \Rightarrow$$

(Using Remark 1 on the rhs)

$$\sum_{\mathbf{n}\in\mathbb{Z}^3}\int_{\mathbb{R}^3}[\ -i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(\mathbf{x})\ +(\sigma a^0)(\mathbf{x})]\ (\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k}+\mathbf{n}+\mathbf{x})d^3\mathbf{x}$$

$$= \int_{\mathbb{R}^3}\partial_1\partial_2\partial_3\Big([i\sum_{\mu=1}^{3}k'_\mu\delta_\mu - \kappa\gamma^0]\hat{\Psi}(\mathbf{k}')\Big)d^3\mathbf{k}'$$

Performing integration on the rhs of the last equation ,bcause of condition 2.4,weobtain that the value of the integral is 0.

Consequently the last equation becomes:

$$\sum_{\mathbf{n}\in\mathbb{Z}^3}\int_{\mathbb{R}^3}[-i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(\mathbf{x})+(\sigma a^0)(\mathbf{x})](\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k}+\mathbf{n}+\mathbf{x})d^3\mathbf{x} = 0 \qquad 4.3$$

Multiplying (16) with $\exp(-i\mathbf{k}\mathbf{s})$ and integrting on $\mathbb{R}^3$,we have:

$$\int_{\mathbb{R}^3}\Big(\sum_{\mathbf{n}\in\mathbb{Z}^3}\int_{\mathbb{R}^3}[-i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(\mathbf{x})+(\sigma a^0)(\mathbf{x})]$$

$$\times(\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k}+\mathbf{n}+\mathbf{x})d^3\mathbf{x}\Big)\exp(-i\mathbf{k}\mathbf{s})d^3\mathbf{k} = 0$$

Interchanging the outer integral with the sum we obtain (see Appendix):

$$\sum_{\mathbf{n}\in\mathbb{Z}^3}\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}[-i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(\mathbf{x})+(\sigma a^0)(\mathbf{x})]$$

$$\times(\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k}+\mathbf{n}+\mathbf{x})d^3\mathbf{x}\Big)\exp(-\mathbf{k}\mathbf{s})d^3\mathbf{k} = 0.$$

Applying the substitution $\mathbf{n}+\mathbf{x}=-\mathbf{r}$ in each inner integral,we obtain the following form:

$$\sum_{\mathbf{n}\in\mathbb{Z}^3}\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}[-i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(-\mathbf{n}-\mathbf{r})+(\sigma a^0)(-\mathbf{n}-\mathbf{r})]$$

$$\times(\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k}-\mathbf{r})d^3\mathbf{r}\Big)\exp(-\mathbf{k}\mathbf{s})d^3\mathbf{k}=0$$

The convolution theorem used in every term of the sum,we obtain:

$$\sum_{\mathbf{n}\in\mathbb{Z}^3}\int_{\mathbb{R}^3}[-i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(-\mathbf{n}-\mathbf{k})+(\sigma a^0)(-\mathbf{n}-\mathbf{k})]\exp(-i\mathbf{k}s)d^3\mathbf{k}$$
$$\times\int_{\mathbb{R}^3}(\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k})\exp(-i\mathbf{k}s)d^3\mathbf{k} = 0.$$

Introducing in every lhs factor of each term of the sum,the substitution $-\mathbf{n}-\mathbf{k}=\mathbf{t}$,there holds true:

$$\sum_{\mathbf{n}\in\mathbb{Z}^3}[\exp(i\mathbf{n}s)-1]\int_{\mathbb{R}^3}[-i\sum_{\mu=1}^{3}\delta_\mu(\sigma a^\mu)(\mathbf{t})+(\sigma a^0)(\mathbf{t})]\exp(i\mathbf{t}s)d^3\mathbf{t}$$
$$\times\int_{\mathbb{R}^3}(\partial_1\partial_2\partial_3\hat{\Psi})(\mathbf{k})\exp(-i\mathbf{k}s)d^3\mathbf{k} = 0.$$

Because of relations 3.5-3.6 of proposition I we immediately obtain relation 2.5 .

β) The continuity of a^μ,Ψ,and the relation 2.5 imply that:

$$\{\sum_{\mu=1}^{3}\delta_\mu a^\mu + ia^0\}\Psi = 0 \qquad 4.5$$

From equations 2.1 and 4.5 we immediately get:

$$(\sum_{\mu=1}^{3}\delta_\mu\partial_\mu + \kappa\gamma^0)\Psi = 0.$$

Inserting in the last relation

$$\Psi = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\hat{\Psi}(\mathbf{k})\exp(-i\mathbf{k}\mathbf{x})d^3\mathbf{k} ,$$

we obtain

$$\int_{\mathbb{R}^3}(\sum_{\mu=1}^{3}-ik_\mu\delta_\mu + \kappa\gamma^0)\hat{\Psi}(\mathbf{k})\exp(-i\mathbf{k}\mathbf{x})d^3\mathbf{k} = 0 \qquad 4.8$$

According to the pressupositions of the theorem,it is evident that the components of $(\sum_{\mu=1}^{3}-ik_\mu\delta_\mu + \kappa\gamma^0)\hat{\Psi}(\mathbf{k})$ are elements of $\mathbb{L}^2$.

Thus,from relation 4.8 we have:

$$(\sum_{\mu=1}^{3}-ik_\mu\delta_\mu + \kappa\gamma^0)\hat{\Psi}(\mathbf{k}) = 0$$

$$\Rightarrow \left(\sum_{\mu=1}^{3} ik_\mu \delta_\mu - \kappa\gamma^0 \right)\left(\sum_{\mu=1}^{3} -ik_\mu \delta_\mu + \kappa\gamma^0 \right)\hat{\Psi}(\mathbf{k}) = 0$$

Performing the multiplication and using the relations

$$\delta_\mu\delta_\nu + \delta_\nu\delta_\mu = -\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu = 0, \text{if } \mu \neq \nu;\ \delta_\mu\delta_\mu = -1\ ;\ \delta_\mu\gamma^0 + \gamma^0\delta_\mu = 0,$$

we have:

$$\left(\sum_{\mu=1}^{3} k_\mu^2 - \kappa^2 \right)\hat{\Psi} = 0$$

Considering that spinor $\hat{\Psi}$ is continuous in $\mathbb{R}^3$, it follows from the last equation that:

$$\hat{\Psi} = 0, \quad \Rightarrow \quad \Psi = 0$$

Corollary:

If the components of the solution Ψ of eq.2.1 are elements of the *S*-space of Schwartz, then $\Psi = 0$.

Remark 6

If the solution Ψ of 2.1 does not obey the constraints 2.3-2.4 of theorem I but it satisfy the conditions of proposition I, then Ψ can be different from zero.

APPENDIX

a) We give here the proof for the possibility to interchange the integration and the summation in the expressions after 3.2.

$$\int_{\mathbb{R}^3} \left(\sum_{\mathbf{n}\in\mathbb{Z}^3} \{ i \sum_{\mu=1}^{3} \delta_\mu a_{\mathbf{n}}^\mu - a_{\mathbf{n}}^0 \} \right)\hat{\Psi}(\mathbf{k})\exp(i(\mathbf{n}-\mathbf{k})\mathbf{x})d^3\mathbf{k} =$$

$$= \sum_{\mathbf{n}\in\mathbb{Z}^3} \left(i\sum_{\mu=1}^{3} a_{\mathbf{n}}^\mu\delta_\mu - a_{\mathbf{n}}^0 \right) \int_{\mathbb{R}^3} \exp(i(\mathbf{n}-\mathbf{k})\mathbf{x})\hat{\Psi}(\mathbf{k})d_3\mathbf{k}.$$

Proof:

Let $b_{\mathbf{n}}^1, b_{\mathbf{n}}^2, b_{\mathbf{n}}^3, b_{\mathbf{n}}^4$ be the components of Spinor

$$\Phi_{\mathbf{n}} := \left(\sum_{\mathbf{n}\in\mathbb{Z}^3} \{ i \sum_{\mu=1}^{3} \delta_\mu a_{\mathbf{n}}^\mu - a_{\mathbf{n}}^0 \} \right)\hat{\Psi}(\mathbf{k})$$

According to a well known theorem of integral calculus, the integral of the first part willbe commutativ with the sum,if ishow that:

$$\sum_{\mathbf{n}\in\mathbb{Z}^3} |b^v_{\mathbf{n}}\exp(i(\mathbf{n}-\mathbf{k})x)| < \infty \quad \text{δηλαδή} \quad \sum_{\mathbf{n}\in\mathbb{Z}^3} |b^v_{\mathbf{n}}| < \infty \quad \text{για } v=1,2,3,4.$$

I will show the above eniquality only for v=1,since the proof for v=2,3,4 is exactly identical: Performing the arithmetic evaluation I find that the component $b^1_{\mathbf{n}}$ of $\Phi_{\mathbf{n}}$is

$$b^1_{\mathbf{n}} = -a^1_{\mathbf{n}}\hat{\psi}_4 + ia^2_{\mathbf{n}}\hat{\psi}_4 - a^3_{\mathbf{n}}\hat{\psi}_3 - a^0_{\mathbf{n}}\hat{\psi}_1$$

$$\Rightarrow \sum_{\mathbf{n}\in\mathbb{Z}^3} |b^1_{\mathbf{n}}| \le |\hat{\psi}_4| \sum_{\mathbf{n}\in\mathbb{Z}^3} |a^1_{\mathbf{n}}| + |\hat{\psi}_4| \sum_{\mathbf{n}\in\mathbb{Z}^3} |a^2_{\mathbf{n}}|$$

$$+ |\hat{\psi}_3| \sum_{\mathbf{n}\in\mathbb{Z}^3} |a^3_{\mathbf{n}}| + |\hat{\psi}_1| \sum_{\mathbf{n}\in\mathbb{Z}^3} |a^0_{\mathbf{n}}|.$$

With the pressuposition of the proposition: $\hat{\psi}_j$ are bounded in $\mathbb{R}^3$ and $a^\mu \in \mathbb{F}^1_{(2n,3)}$that is $\sum |a^\mu_{\mathbf{n}}| < \infty$,we have $\sum |b^1_{\mathbf{n}}| < \infty$.

β) The commutation of the integral with the sum in the relation

$$\sum_{\mathbf{n}\in\mathbb{Z}^3} \int_{\mathbb{R}^3} \Big(\sum_{\mu=1}^{3} ia^\mu_{\mathbf{n}}\delta_\mu - a^0_{\mathbf{n}} \Big)\hat{\psi}(\mathbf{n}+\mathbf{k}')\exp(-i\mathbf{k}'\mathbf{x})d^3\mathbf{k}'$$

is proven identically.

γ) It will be shown that the integral of the following expression commutes with the sum

$$\int_{\mathbb{R}^3} \Big(\sum_{\mathbf{n}\in\mathbb{Z}^3} \int_{\mathbb{R}^3} [-i\sum_{\mu=1}^{3} \delta_\mu(\sigma a^\mu)(\mathbf{x}) + (\sigma a^0)(\mathbf{x})]$$
$$\times(\partial_1\partial_2\partial_3\hat{\psi})(\mathbf{k}+\mathbf{n}+\mathbf{x})d^3\mathbf{x}\Big)\exp(-i\mathbf{k}s)d^3\mathbf{k}.$$

Proof:

Let $A^1_{\mathbf{n}},A^2_{\mathbf{n}},A^3_{\mathbf{n}},A^4_{\mathbf{n}}$ be the components of

$$\Big(\sum_{\mathbf{n}\in\mathbb{Z}^3} \int_{\mathbb{R}^3} [-i\sum_{\mu=1}^{3} \delta_\mu(\sigma a^\mu)(\mathbf{x}) + (\sigma a^0)(\mathbf{x})](\partial_1\partial_2\partial_3\hat{\psi})(\mathbf{k}+\mathbf{n}+\mathbf{x})d^3\mathbf{x}\Big)$$

and $B^1_{\mathbf{n}}, B^2_{\mathbf{n}}, B^3_{\mathbf{n}}, B^4_{\mathbf{n}}$ the components of

$$I = \int_{\mathbf{k+n}}^{\mathbf{k+n+1}} \partial_1\partial_2\partial_3\Big([i\sum_{\mu=1}^{3} k'_\mu\delta_\mu - \kappa\gamma^0]\Psi(\hat{\mathbf{k}}')\Big)d^3\mathbf{k}'$$

The commutation of the sum with the integral is possible, if we can prove that

$$\sum_{\mathbf{n}\in\mathbf{Z}^3} |A^v_{\mathbf{n}}\exp(-i\mathbf{k}\mathbf{s})| < \infty$$

$$\Rightarrow \sum_{\mathbf{n}\in\mathbf{Z}^3} |A^v_{\mathbf{n}}| < \infty \quad \text{for } v=1,2,3,4 :$$

It is sufficient to show the last inequality for v=1 (since for v=2,3,4 the proof is identical)

From relation 4.2 of the theorem we obtain:

$$A^1_{\mathbf{n}} = B^1_{\mathbf{n}} \quad \text{for} \quad \mathbf{n} \in \mathbf{Z}^3 \qquad \text{A1}$$

Evaluating the integral I we obtain:

$$B^1_{\mathbf{n}} = \int_{\mathbf{k+n}}^{\mathbf{k+n+1}} \{-\partial_1\partial_2\partial_3(k'_1\hat{\Psi}_4) + i\partial_1\partial_2\partial_3(k'_2\hat{\Psi}_4) - \partial_1\partial_2\partial_3(k'_3\hat{\Psi}_3) - \kappa\partial_1\partial_2\partial_3\hat{\Psi}_1\}d^3\mathbf{k}' \qquad \text{A2}$$

Substituting (20) in (19) and suming, we have;

$$\sum_{\mathbf{n}\in\mathbf{Z}^3} |A^1_{\mathbf{n}}| = \sum_{\mathbf{n}\in\mathbf{Z}^3} \Big| \int_{\mathbf{k+n}}^{\mathbf{k+n+1}} \{-\partial_1\partial_2\partial_3(k'_1\hat{\Psi}_4) + i\partial_1\partial_2\partial_3(k'_2\hat{\Psi}_4) - \partial_1\partial_2\partial_3(k'_3\hat{\Psi}_3) - \kappa\partial_1\partial_2\partial_3\hat{\Psi}_1\}d^3\mathbf{k}' \Big| \le$$

$$\le \sum_{\mathbf{n}\in\mathbf{Z}^3} \int_{\mathbf{k+n}}^{\mathbf{k+n+1}} \{|\partial_1\partial_2\partial_3(k'_1\hat{\Psi}_4)| + |\partial_1\partial_2\partial_3(k'_2\hat{\Psi}_4)| + |\partial_1\partial_2\partial_3(k'_3\hat{\Psi}_3)| + |\kappa\partial_1\partial_2\partial_3\hat{\Psi}_1|\}d^3\mathbf{k}'$$

(because of Remark 1)

$$= \int_{\mathbb{R}^3} \{ |\partial_1\partial_2\partial_3(k_1'\hat{\Psi}_4)| + |\partial_1\partial_2\partial_3(k_2'\hat{\Psi}_4)| + |\partial_1\partial_2\partial_3(k_3'\hat{\Psi}_3)|$$

$$+ |\kappa\partial_1\partial_2\partial_3\hat{\Psi}_1| \} d^3\mathbf{k}' < \infty$$

due to the condition 2.3 of the theorem.

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