Application of the quantum mechanical hypervirial theorems to even-power series potentials

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The class of the even-power series potentials:$V(r) = -D + \sum_{k=0}^{\infty} V_k \lambda^k r^{2k+2}$, $V_0 = \omega^2 > 0$, is studied with the aim of obtaining approximate analytic expressions for the energy eigenvalues, the expectation values for the potential and the kinetic energy operator, and the mean square radii of the orbits of a particle in its ground and excited states. We use the Hypervirial Theorems (HVT) in conjunction with the Hellmann-Feynman Theorem (HFT) which provide a very powerful scheme especially for the treatment of that type of potentials, as previous studies have shown. The formalism is reviewed and the expressions of the above mentioned quantities are subsequently given in a convenient way in terms of the potential parameters and the mass of the particle, and are then applied to the case of the Gaussian potential and to the potential $V(r) = \frac{-D}{\cosh^2(\frac{r}{R})}$. These expressions are given in the form of series expansions, the first terms of which yield in quite a number of cases values of very satisfactory accuracy.

I. INTRODUCTION

Various approaches can be employed in an effort to obtain approximate expressions for the energy eigenvalues and other quantities of interest for a particle moving in a certain potential. Unfortunately the majority entails cumbersome calculations of matrix elements or the ingenious deduction of a trial function. The previous obstacles can be overcome by using the Hypervirial Theorem[1] in conjunction with the Hellmann-Feynman theorem, the "HVT-HFT" method, which could have been labeled as classical, had it not been for the peculiarities of the potential we select each time. Consequently, this method constitutes a very effective alternative to a pure perturbation or variational treatment.

A multitude of authors has applied variations of this very method to a good number of potentials[2-10]. The results vary each time, according to the susceptibility of the potential in question. The scheme makes use of a perturbation parameter λ which enables us to expand the potential as well as the energy and the moments in a λ-series. We traditionally adopt the first non trivial term of our expanded potential as the unperturbed term of our problem. The energy of the unperturbed Schroedinger equation will serve as a parameter, in terms of which the other terms of the series will be calculated.

In the present work we consider a wide class of potentials of the form $V(r) = -D + \sum\limits_{k=0}^{\infty} V_k \lambda^k r^{2k+2}, V_0 = \omega^2 > 0$. The general formalism for such a class of potentials is reviewed in the next section and subsequently a general approximate expression for the energy eigenvalues is derived in terms of the potential parameters and the mass of the particle. In Sec. III, the corresponding general expressions for the expectation values of the kinetic and the potential energy operator are derived, and also for the mean square radius of the orbit of a particle in any bound state in the potential.

In Sec. V and Sec. VI, the derived expressions are applied to the well known Gaussian potential and to the Poeschl-Teller(P.T)-type potential $V(r) = \frac{-D}{\cosh^2(\frac{r}{R})}$, respectively. The final section is devoted to a test of the accuracy of the derived expressions by comparing their yielded values, mainly with the corresponding results obtained with a numerical integration of the Schroedinger equation. That is accomplished by taking as an example the non-relativistic motion of a $\Lambda-$particle in a hypernucleus moving in a Gaussian or in the above-mentioned P-T potential.

II. DESCRIPTION OF THE "HVT-HFT" SCHEME FOR EVEN-POWER POTENTIALS. DERIVATION OF A GENERAL APPROXIMATE EXPRESSION FOR THE ENERGY EIGENVALUES.

We consider a class of potentials $V(r)$, which can be written in the form

$$V(r) = -D + \sum_{k=0}^{\infty} V_k \lambda^k r^{2k+2} \tag{1}$$

where $V_0 = \omega^2 > 0$. It is obvious that certain anharmonic oscillators, like for example the $V(r) = \frac{1}{2}kr^2 + k_1 r^m$, which was studied in Ref. 3,(see also p.339 of Ref.2a and references therein) belongs in that class. Furthermore,in that class, there belongs a number of potentials encountered in applications. Those potentials are of the form:

$$V(r) = -Df(\frac{r}{R}) \qquad 0 \leq r < \infty \tag{2}$$

where $D > 0$ is the potential depth, $R > 0$ the potential radius and the "potential form factor" f, $(f(0) = 1)$, is an even analytic function of $x = \frac{r}{R}$. Namely,

$$f(x) = \sum_{k=0}^{\infty} d_k x^{2k} \tag{3}$$

where d_k are the numbers:

$$d_k = \frac{1}{(2k)!} \frac{d^{2k}}{dx^{2k}} f(x) \mid_{x=0} \qquad k = 0, 1, 2, ... \tag{4}$$

The radial Schrödinger equation, for the radial wave function $u_{nl} = rR_{nl}(r)$, then reads:

$$\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + \sum_{k=0}^{\infty} V_k \lambda^k r^{2k+2} \right] u_{nl} = \tilde{E}_{nl} \, u_{nl} \tag{5}$$

where $\tilde{E}_{nl}$ is the shifted energy eigenvalue E_{nl}. That is

$$\tilde{E}_{nl} = E_{nl} + D \tag{6}$$

Setting

$$\tilde{V}(r) = \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + \sum_{k=0}^{\infty} V_k \lambda^k r^{2k+2} \tag{7}$$

the Hamiltonian becomes

$$\tilde{H} = -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \tilde{V}(r) \tag{8}$$

As a result the Schroedinger equation can now be written as:

$$\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \tilde{V}(r) \right] u_{nl} = \tilde{E}_{nl}\, u_{nl} \tag{9}$$

The Hamiltonian $\tilde{H}$ is now identical to the one used by Swenson and Danford[3], which means that we can make full use of the Hypervirial relation that they obtained. Namely:

$$\tilde{E}_{nl} < r^N >_{nl} = \frac{1}{2}(N+1)^{-1} < r^{N+1} \frac{d\tilde{V}}{dr} >_{nl} + < r^N \tilde{V} >_{nl} - \frac{\hbar^2}{8\mu} N(N-1) < r^{N-2} >_{nl} \tag{10}$$

If we expand the energy $\tilde{E}_{nl}$ and the moments $< r^N >$ in λ-series we have respectively:

$$\tilde{E}_{nl} = \sum_{k=0}^{\infty} E^{(k)} \lambda^k \tag{11}$$

$$< r^N >_{nl} = \sum_{k=0}^{\infty} C_N^{(k)} \lambda^k \tag{12}$$

Note that the indices (nl) on $E^{(k)}$ and $C_{N+2}^{(k)}$ (which follows) will be suppressed for simplicity. According to the above definitions the unperturbed term of our potential is

$$\tilde{V}^{(0)}(r) = \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + \omega^2 r^2 \tag{13}$$

and the corresponding energy eigenvalue is

$$E^{(0)} = 2\omega a_{nl} \sqrt{\frac{\hbar^2}{2\mu}} \tag{14}$$

which is, in fact, the energy of the Harmonic Oscillator. In the interest of elegance we have adopted the following notation:

$$a_{nl} = (2n + l + \frac{3}{2})$$ (15)

$n = 0, 1, 2, 3, ...$(principal quantum number)
$l = 0, 1, 2, 3, ...$(orbital quantum number)
Applying the Hellmann-Feynman theorem(HFT)[15,16] to our Hamiltonian we get:

$$\frac{\partial \tilde{E}_{nl}}{\partial \lambda} = < \frac{\partial \tilde{H}}{\partial \lambda} >$$ (16)

After some trivial calculations we obtain:

$$E^{(k)} = \frac{1}{k} \sum_{m=0}^{k} m V_m C_{2m+2}^{(k-m)}$$ (17)

Substituting (8) and (9) into (7) we get the familiar recurrence relation[6]:

$$C_{N+2}^{(k)} = \frac{N+1}{(N+2)\omega^2} \left\{ \sum_{q=0}^{k} E^{(q)} C_N^{(k-q)} - \left[\frac{Nl(l+1)}{N+1} - \frac{N(N-1)}{4} \right] \frac{\hbar^2}{2\mu} C_{N-2}^{(k)} - \right.$$

$$\left. - \frac{N+3}{N+3} V_1 C_{N+4}^{(k-1)} - \frac{N+4}{N+1} V_2 C_{N+6}^{(k-2)} - ... - \frac{N+k+2}{N+1} V_k C_{N+k+2}^{(0)} \right\}$$ (18)

where $C_0^{(k)} = \delta_{0k}$.

The recurrence relation (18) is used to evaluate the coefficients $E^{(k)}$ used by (11). In fact what we need to do is evaluate the coefficients $C_N^{(k)}$used by (12) and (17). We perform this calculation in a hierarchical manner[8] i.e., To obtain $E^{(1)}$ in (11) we resort to (17) which readily yields

$$E^{(1)} = V_1 C_4^{(0)}$$ (19)

We still need to calculate the $C_4^{(0)}$ coefficient which will be extracted from (18). Indeed, (18) gives

$$C_4^{(0)} = E^{(0)} C_2^{(0)}$$ (20)

To calculate $C_2^{(0)}$ we need to resort to (18) once more so that we get

$$C_2^{(0)} = \frac{E^{(0)}}{2\omega^2}$$ (21)

$E^{(1)}$ can be obtained by virtue of (19), (20) and (21).

The same pattern of calculations is adhered to, for each coefficient (either $E^{(k)}$ or $C_N^{(k)}$). Applying this very method to the class of potentials given by (1) we arrive at the following general expression (where $b = \frac{\hbar^2}{2\mu}$):

$$E_{nl} = -D + E^{(0)} + E^{(1)}\lambda + E^{(2)}\lambda^2 + E^{(3)}\lambda^3 + E^{(4)}\lambda^4 + ... \qquad (22)$$

$$= -D + 2\omega a_{nl}b^{\frac{1}{2}} + \frac{V_1}{8\omega^2}\left[12a_{nl}^2 - 4l(l+1) + 3\right]b\lambda -$$

$$-\frac{a_{nl}}{32\omega^5}[4a_{nl}^2(17V_1^2 - 20V_2\omega^2) + 12l(l+1)(4V_2\omega^2 - 3V_1^2) + 67V_1^2 - 100V_2\omega^2]b^{\frac{3}{2}}\lambda^2 +$$

$$+\frac{1}{1024\omega^8}\left\{80a_{nl}^4(75V_1^3 - 132V_1V_2\omega^2 + 56V_3\omega^4) - 8a_{nl}^2[12l(l+1)(43V_1^3 - 84V_1V_2\omega^2 +\right.$$

$$+40V_3\omega^4) - 1707V_1^3 + 20\omega^2(177V_1V_2 - 98V_3\omega^2)] + [4l(l+1)(11V_1^3 - 36V_1V_2\omega^2 +$$

$$+24V_3\omega^4) - 3(171V_1^3 - 420V_1V_2\omega^2 + 280V_3\omega^4)][4l(l+1) - 3]\right\}b^2\lambda^3 -$$

$$-\frac{a_{nl}}{8192\omega^{11}}\left\{48a_{nl}^4[3563V_1^4 - 8344V_1^2V_2\omega^2 + 4032V_1V_3\omega^4 + 16\omega^4(131V_2^2 - 84V_4\omega^2)]\right.$$

$$-40a_{nl}^2[4l(l+1)(891V_1^4 - 2280V_1^2V_2\omega^2 + 1216V_1V_3\omega^4 + 642V_2^2\omega^4 - 448V_4\omega^6)$$

$$-17833V_1^4 + 47288V_1^2V_2\omega^2 - 28224V_1V_3\omega^4 - 13264V_2^2\omega^4 + 12096V_4\omega^6]$$

$$+48[l(l+1)^2][303V_1^4 - 1048V_1^2V_2\omega^2 + 704V_1V_3\omega^4 + 16\omega^4(23V_2^2 - 20V_4\omega^2)]$$

$$-8l(l+1)(28647V_1^4 - 85656V_1^2V_2\omega^2 + 58048V_1V_3\omega^4 + 27120V_2^2\omega^4 - 28480V_4\omega^6) +$$

$$305141V_1^4 - 938248V_1^2V_2\omega^2 + 673344V_1V_3\omega^4 + 16\omega^4(19277V_2^2 - 22428V_4\omega^2)\left.\right\}b^{\frac{5}{2}}\lambda^4 + ...$$
$$(23)$$

For potentials of the general form (2), the above expression may be rewritten in terms of the depth and radius parameters. The relation which links the coefficients V_k with the coefficients d_k is the following (for $\lambda = \frac{1}{R^2}$):

$$V_k = -\frac{D}{R^2}d_{k+1} \qquad k = 0, 1, 2, 3, ... \qquad (24)$$

Thus, in that case, the expansion for the energy eigenvalue becomes :

$$E = -D + \frac{2a_{nl}(-Dd_1)^{\frac{1}{2}}}{R}b^{\frac{1}{2}} + \frac{d_2}{8d_1R^2}\left[12a_{nl}^2 - 4l(l+1) + 3\right]b-$$

$$-\frac{a_{nl}(-Dd_1)^{\frac{1}{2}}}{32Dd_1^3R^3}\left\{4a_{nl}^2\left(20d_1d_3 - 17d_2^2\right) + 4d_1d_3\left[25 - 12l(l+1)\right] + d_2^2\left[36l(l+1) - 67\right]\right\}b^{\frac{3}{2}}-$$

$$-\frac{1}{1024Dd_1^4R^4}\left\{80a_{nl}^2\left(56d_1^2d_4 - 132d_1d_2d_3 + 75d_2^3\right) - 8a^2\{40d_1^2d_4\left[12l(l+1) - 49\right]+\right.$$

$$+12d_1d_2d_3\left[295 - 84l(l+1)\right] + 3d_2^3\left[172l(l+1) - 569\right]\} + \left[4l(l+1) - 3\right] \times$$

$$\times\left\{24d_1^2d_4\left[4l(l+1) - 35\right] + 36d_1d_2d_3\left[35 - 4l(l+1)\right] + d_2^3\left[44l(l+1) - 513\right]\right\}b^2+...$$

$$(25)$$

It is of interest that Eq. (25) coincides with the corresponding one of Ref. 13
(If correction of a few misprints or minor errors is made[14]). That corroborates
the validity of the above expressions for E_{nl}.

Eq. (25) is suitable to examine under which conditions the first terms of
the expansion lead to a good approximation for the energy eigenvalues. By
considering the ratio of successive terms we easily realise that the above
desirable condition is satisfied when the parameter $s^{-1} = \left(\frac{\hbar^2}{2\mu DR^2}\right)^{\frac{1}{2}}$ is small
that is when the potential well is quite deep and wide.

According to the previous relations the expression for the lowest energy
level spacing $(\Delta E)_L = E_{nl}(n = 0, l = 1) - E_{nl}(n = 0, l = 0)$ is easily derived.
The general form has been calculated up to the third order (λ^3), and is as
follows:

$$(\Delta E)_L = E_{1p} - E_{1s} = 2\omega b^{\frac{1}{2}} + \frac{5V_1}{\omega^2}\lambda b - \frac{15}{4\omega^5}\left(6V_1^2 - 7V_2\omega^2\right)\lambda^2 b^{\frac{3}{2}}+$$

$$+\frac{5}{32\omega^8}\left(1271V_1^3 - 2268V_1V_2\omega^2 + 1008V_3\omega^4\right)\lambda^3 b^2 - ... \qquad (26)$$

For potentials of the form (2) one can also write the relation above in terms
of the depth and radius parameters.

$$(\Delta E)_L = \frac{2(-Dd_1)^{\frac{1}{2}}}{R}b^{\frac{1}{2}} + \frac{5d_2}{2d_1R^2}b - \frac{15(-Dd_1)^{\frac{1}{2}}}{4Dd_1^3R^3}\left(7d_1d_3 - 6d_2^2\right)b^{\frac{3}{2}}$$

$$-\frac{5}{32Dd_1^4R^4}\left(1008d_1^2d_4 - 2268d_1d_2d_3 + 1271d_2^3\right)b^2 - ... \qquad (27)$$

III. APPROXIMATE EXPRESSIONS FOR THE $<V>, <T>$ AND $<r^2>$ IN TERMS OF THE POTENTIAL PARAMETERS.

A great number of authors has applied the Hypervirial relations to specific potentials only to obtain an asymptotic series for the energy eigenvalue, the general expression of which has been derived in the previous section for the class of the potentials considered. Nevertheless, the matter of the expectation values (with respect to the energy eigenfunctions) of the kinetic energy operator: $<T>_{nl}=<T>$ and of the potential energy operator $<V>_{nl}=<V>$, as well as of the mean square radius of the orbit of a particle in a given energy eigenstate: $<r^2>_{nl}=<r^2>$, remains to be settled. Below we extract general expressions for those quantities.

As for the expectation value for the kinetic energy operator, the application of the Hellmann-Feynman theorem[15,16], according to which:

$$\frac{<T>}{\mu} = -\frac{\partial E}{\partial \mu} \tag{28}$$

yields a λ-series of the form:

$$<T> = \sum_{k=0}^{\infty} T^{(k)} \lambda^k \tag{29}$$

The relation between $T^{(k)}$ and $E^{(k)}$ can be easily extracted:
The $E^{(k)}$ terms of the energy-series can be written as $E^{(k)} = e^{(k)} b^{\frac{k+1}{2}}$ where $e^{(k)}$ is independent of the mass μ. Therefore the energy series reads:

$$E = -D + \sum_{k=0}^{\infty} e^{(k)} b^{\frac{k+1}{2}} \lambda^k \tag{30}$$

Differentiating the energy series with respect to μ one obtains:

$$\frac{\partial E}{\partial \mu} = \frac{\partial b}{\partial \mu} \frac{\partial E}{\partial b} = \left(\frac{\partial b}{\partial \mu}\right) \sum_{k=0}^{\infty} \left(\frac{k+1}{2}\right) e^{(k)} b^{\frac{k-1}{2}} \lambda^k \tag{31}$$

Since $\frac{\partial b}{\partial \mu} = -\frac{b}{\mu}$, Eq. (31) can now be written as:

$$\frac{\partial E}{\partial \mu} = \left(-\frac{b}{\mu}\right) \sum_{k=0}^{\infty} \left(\frac{k+1}{2}\right) e^{(k)} b^{\frac{k-1}{2}} \lambda^k \tag{32}$$

Eventually we obtain the relation:

$$\frac{\partial E}{\partial \mu} = \left(-\frac{1}{\mu}\right) \sum_{k=0}^{\infty} \left(\frac{k+1}{2}\right) e^{(k)} b^{\frac{k+1}{2}} \lambda^k \tag{33}$$

By a simple inspection of (28) and (33), we can see that the coefficients for the kinetic and total energy are related through the formula:

$$T^{(k)} = \left(\frac{k+1}{2}\right) E^{(k)} \tag{34}$$

The expressions of $E^{(k)}$ follow immediately from expressions (22) and (23), [or (25)].

On the other hand, since $E = < T > + < V >$, we can readily extract a similar λ-series for the expectation value for of the potential energy operator: This is obviously in the form:

$$< V > = -D + \sum_{k=0}^{\infty} V^{(k)} \lambda^k \tag{35}$$

where

$$V^{(k)} = \left(\frac{1-k}{2}\right) E^{(k)} \tag{36}$$

By using relations (17) and (18) we can extract the coefficients $C_N^{(k)}$ used by (12) in order to evaluate the orbital mean square radii. The λ-series for the $< r^2 >$ is:

$$< r^2 >_{nl} = C_2^{(0)} + C_2^{(1)} \lambda + C_2^{(2)} \lambda^2 + C_2^{(3)} \lambda^3 + \ldots$$

$$= \frac{a_{nl}}{\omega} b^{\frac{1}{2}} - \frac{V_1}{8\omega^4} [12a_{nl}^2 - 4l(l+1) + 3] b\lambda +$$

$$+ \frac{a_{nl}}{64\omega^7} [20a_{nl}^2(17V_1^2 - 12V_2\omega^2) + 36l(l+1)(4V_2\omega^2 - 5V_1^2) + 355V_1^2 - 300V_2\omega^2] b^{\frac{3}{2}} \lambda^2 -$$

$$- \frac{1}{256\omega^{10}} \left\{ 8a_{nl}^4 (75V_1^3 - 99V_1 V_1\omega^2 + 28V_3\omega^4) - 8a_{nl}^2 12l(l+1)(43V_1^3 - 63V_1 V_2\omega^2 + \right.$$

$$+ 20V_3\omega^4) - 1707V_1^3 + 2655V_1 V_2\omega^2 - 980V_3\omega^4) + 4l(l+1)(11V_1^3 - 27V_1 V_2\omega^2 +$$

$$\left. + 12V_3\omega^4) - 3(171V_1^3 - 315V_1 V_2\omega^2 + 140V_3\omega^4 [4l(l+1) - 3] \right\} b^2 \lambda^3 + \ldots \tag{37}$$

For potentials of the form(2) the $< r^2 >_{nl}$ is expressed in terms of the depth and radius as follows:

$$< r^2 >_{nl} = \frac{a_{nl}R}{(-Dd_1)^{\frac{1}{2}}} b^{\frac{1}{2}} + \frac{d_2}{8Dd_1^2} \left[12a_{nl}^2 - 4l(l+1) + 3\right] b-$$

$$-\frac{a_{nl}(-Dd_1)^{\frac{1}{2}}}{64D^2d_1^4R} \left\{20a_{nl}^2\left(12d_1d_3 - 17d_2^2\right) + 12d_1d_3\left[25 - 12l(l+1)\right] + 5d_2^2\left[36l(l+1) - 67\right]\right\} b^{\frac{3}{2}}-$$

$$-\frac{1}{256D^2d_1^5R^2} \left\{80a_{nl}^4\left(28d_1^2d_4 - 99d_1d_2d_3 + 75d_2^3\right) - 8a_{nl}^2\{20d_1^2d_4\left[12l(l+1) - 49\right]+\right.$$

$$+9d_1d_2d_3\left[295 - 84l(l+1)\right] + 3d_2^3\left[172l(l+1) - 569\right]\} + \left[4l(l+1) - 3\right] \times$$

$$\times\left\{12d_1^2d_4\left[4l(l+1) - 35\right] + 27d_1d_2d_3\left[35 - 4l(l+1)\right] + d_2^3\left[44l(l+1) - 513\right]\right\} b^2+...$$
$$(38)$$

The same way we can obtain the moments of higher order.

IV. APPLICATION OF THE "HVT-HFT" SCHEME
TO THE GAUSSIAN POTENTIAL.

A member of the potential class in question is the Gaussian potential, which is given by the formula:

$$V(r) = -De^{\frac{-r^2}{R^2}}, \qquad 0 \leq r < \infty \qquad (39)$$

where D is the potential depth and R its radius. Such a potential as the one discussed in the following section may be considered for example a first approximation to the self-consistent potential for a nucleon in light nuclei, or for a $\Lambda-$hyperon in light hypernuclei in non-relativistic Quantum Mechanics. In such a case the potential radius R is expressed (in the framework of the rigid core model) in terms of the mass number of the core nucleus by means of the relation $R = r_0A_c^{\frac{1}{3}}$ where A_c is the mass number of the core nucleus ($A_c = A - 1$). The expression for d_k for the Gaussian potential is:

$$d_k = \frac{(-1)^k}{k!} \qquad (40)$$

The energy λ-series for that particular potential has already been obtained by Lai, up to λ^3 [6].

In our study of the Gaussian potential, we not only derive the expectation values of both the potential and the kinetic energy but also provide the m.s.radii of the particle orbits states, $< r^2 >_{nl}$. Moreover, below,the total energy is given up to λ^4.

$$E_{nl} = -D + 2a_{nl}\omega b^{\frac{1}{2}} - \frac{1}{16}[12a_{nl}^2 - 4l(l+1) + 3]b\lambda - \frac{a_{nl}}{384\omega}[44a_{nl}^2 - 12l(l+1) + 1]b^{\frac{3}{2}}\lambda^2 -$$

$$-\frac{1}{24576\omega^2}\left\{1360a_{nl}^4 + 8a_{nl}^2[1 - 12l(l+1)] + 3[3 - 4l(l+1)][20l(l+1) - 47]\right\}b^2\lambda^3 -$$

$$-\frac{a_{nl}}{5898240\omega^3}\left\{208848a_{nl}^4 + 3640a_{nl}^2[12l(l+1) - 1] - 86640[l(l+1)]^2 +\right.$$

$$\left. + 306600l(l+1) - 300383\right\}b^{\frac{5}{2}}\lambda^4 + ... \tag{41}$$

where $\lambda = \frac{1}{R^2}$ and $\omega^2 = \frac{D}{R^2}$
According to (34) the λ-series for the kinetic energy is:

$$< T >_{nl} = a_{nl}\omega b^{\frac{1}{2}} - \frac{1}{16}\left[12a_{nl}^2 - 4l(l+1) + 3\right]b\lambda - \frac{a_{nl}}{256\omega}\left[44a_{nl}^2 - 12l(l+1) + 1\right]b^{\frac{3}{2}}\lambda^2$$

$$-\frac{1}{12288\omega^2}\left\{1360a_{nl}^4 + 8a_{nl}^2[1 - 12l(l+1)] + 3[3 - 4l(l+1)][20l(l+1) - 47]\right\}b^2\lambda^3$$

$$-\frac{a_{nl}}{2359296\omega^3}\left\{208848a_{nl}^4 + 3640a_{nl}^2[12l(l+1) - 1] - 86640[l(l+1)]^2\right.$$

$$\left. + 306600l(l+1) - 300383\right\}b^{\frac{5}{2}}\lambda^4 + ... \tag{42}$$

The corresponding one for the potential energy follows immediately from $< V >_{nl} = E_{nl} - < T >_{nl}$. Finally the expression for $< r^2 >_{nl}$ is:

$$< r^2 >_{nl} = \frac{a_{nl}}{\omega}b^{\frac{1}{2}} + \frac{1}{16\omega}\left[12a_{nl}^2 - 4l(l+1) + 3\right]b\lambda +$$

$$+\frac{3a_{nl}}{256\omega^3}\left[60a_{nl}^2 - 28l(l+1) + 45\right]b^{\frac{3}{2}}\lambda^2 +$$

$$+\frac{1}{6144\omega^4}\left\{4400a_{nl}^4 + 8a_{nl}^2[791 - 276l(l+1)] +\right.$$

$$\left. + 3[3 - 4l(l+1)][12l(l+1) + 23]\right\}b^2\lambda^3 + ... \tag{43}$$

The lowest energy level spacing can be given by (26) and is as follows:

$$(\Delta E)_L = E_{1p} - E_{1s} = 2\omega b^{\frac{1}{2}} - \frac{5}{2}\lambda b - \frac{5}{4\omega}\lambda^2 b^{\frac{3}{2}} - \frac{475}{256\omega^2}\lambda^3 b^2 - ... \qquad (44)$$

V. APPLICATION OF THE "HVT-HFT" SCHEME TO THE POTENTIAL: $\frac{-D}{\cosh^2(\frac{r}{R})}$.

An interesting feature of the potential

$$V(r) = \frac{-D}{\cosh^2(\frac{r}{R})} \qquad 0 \le r < \infty \qquad (45)$$

which is a special case of the well-known Poeschl-Teller potential, is that the corresponding Schroedinger eigenvalue problem can be solved exactly for the s-states. That potential has been used rather extensively in studies of hypernuclei[11]. The exact analytic expression for the s-state energy eigenvalues is [11,12]:

$$E_{n0} = -\frac{\hbar^2}{2\mu R^2}\left[2n + \frac{3}{2} - \frac{1}{2}\sqrt{\frac{8\mu D R^2}{\hbar^2} + 1}\right]^2 \qquad (46)$$

Note that for the energy eigenvalues

$$n = 0, 1, 2, ... \qquad n_{\max} < \frac{1}{4}\left[\left(1 + \frac{8\mu D R^2}{\hbar^2}\right)^{\frac{1}{2}} - 1\right]$$

For the same states exact analytic results are also available ,through the application of the Hellmann-Feynman theorem, for $< T >$ and $< V >$ and thus one can test safely whether the first terms of the corresponding expansions constitute a good approximation to the relevant quantities. The expressions [11] for $< T >_{n0}$ and $< V >_{n0}$ can be written respectively:

$$< T >_{n0} = E_{n0} + D - \frac{2D\left(2n + \frac{3}{2}\right)}{\sqrt{\frac{8\mu D R^2}{\hbar^2} + 1}} \qquad n = 0, 1, 2, ... \qquad (47)$$

$$< V >_{n0} = -D + \frac{2D\left(2n + \frac{3}{2}\right)}{\sqrt{\frac{8\mu DR^2}{\hbar^2} + 1}} \qquad n = 0, 1, 2, \ldots \qquad (48)$$

The expressions for the coefficients d_k are:

$$d_0 = 1 \qquad (49)$$

$$d_1 = -1 \qquad (50)$$

$$d_2 = \frac{2}{3} \qquad (51)$$

$$d_3 = -\frac{17}{45} \qquad (52)$$

$$d_4 = \frac{62}{315} \qquad (53)$$

$$d_5 = -\frac{1382}{14175} \qquad (54)$$

According to the previous analysis we have the following HVT-energy series (again here $\lambda = \frac{1}{R^2}$ and $\omega^2 = \frac{D}{R^2}$):

$$E_{nl} = -D + 2\omega a_{nl} b^{\frac{1}{2}} - \frac{1}{12}\left[12a_{nl}^2 - 4l(l+1) + 3\right] b\lambda + \frac{a_{nl}}{60\omega}\left[15 - 4l(l+1)\right] b^{\frac{3}{2}}\lambda^2 -$$

$$-\frac{4}{945\omega^2}l(l+1)\left[12a_{nl}^2 - 4l(l+1) + 3\right] b^2\lambda^2 - \frac{a_{nl}}{907200\omega^3}\left\{33280a_{nl}^2 l(l+1) - \right.$$

$$-12816[l(l+1)]^2 + 25\left[440l(l+1) + 567\right]\right\} b^{\frac{5}{2}}\lambda^3 - \frac{1}{1247400\omega^4}l(l+1)\left\{32720a_{nl}^4 + \right.$$

$$+8a_{nl}^2\left[1445 - 828l(l+1)\right] + \left[3 - 4l(l+1)\right]\left[1084l(l+1) - 2445\right]\right\} b^3\lambda^4 + \ldots$$

$$(55)$$

If we set $l = 0$ to the above expression, the ensuing expansion is summed up, leading to a closed form result which is identical to relation (46). According to relation (34) we can derive the expectation value of the kinetic energy in the form of a λ-series. Applying the same formalism:

$$< T >_{nl} = a_{nl}\omega b^{\frac{1}{2}} - \frac{1}{12}\left[12a_{nl}^2 - 4l(l+1) + 3\right] b\lambda - \frac{a_{nl}}{40\omega}\left[4l(l+1) - 15\right] b^{\frac{3}{2}}\lambda^2 -$$

$$-\frac{8}{945\omega^2}l(l+1)\left[12a_{nl}^2 - 4l(l+1) + 3\right] b^2\lambda^3 - \frac{a_{nl}}{362880\omega^3}\left\{33280a_{nl}^2 l(l+1) - 12816l^4 - \right.$$

$$-25632l^3 - 1816l^2 + 11000l + 14175 \} b^{\frac{5}{2}}\lambda^4 - \frac{1}{415800\omega^4}l(l+1)\{32720a_{nl}^4 -$$

$$-8a_{nl}^2[828l(l+1) - 1445] - [4l(l+1) - 3][1084l(l+1) - 2445]\} b^3\lambda^4 + ...$$

$$(56)$$

Using Eqs. (37) we can obtain the $< r^2 >$ λ-series:

$$< r^2 > = \frac{a_{nl}}{\omega}b^{\frac{1}{2}} + \frac{1}{12\omega^2}\left[12a_{nl}^2 - 4l(l+1) + 3\right]b\lambda +$$

$$+\frac{a_{nl}}{90\omega^3}\left[85a_{nl}^2 - 36l(l+1) + 50\right]b^{\frac{3}{2}}\lambda^2 + \frac{1}{120960\omega^4}\{105840a_{nl}^4 + 24a_{nl}^2[4725 -$$

$$-1532l(l+1)] + [3 - 4l(l+1)][2228l(l+1) - 1575]\} b^2\lambda^3 + ... \quad (57)$$

Finally, the expression for the lowest energy level spacing becomes:

$$(\Delta E)_L = E_{1p} - E_{1s} = 2\omega b^{\frac{1}{2}} - \frac{90}{27}\lambda b - \frac{1}{12\omega}\lambda^2 b^{\frac{3}{2}} - \frac{16}{27\omega^2}\lambda^3 b^{\frac{3}{2}} - ... \quad (58)$$

VI. NUMERICAL RESULTS AND COMMENTS.

In this section we give numerical results for the Gaussian potential as well as for the Poeschl-Teller-type one (45). The potential parameters we use here for purposes of illustration were obtained by a least squares fitting procedure to experimental 1s-state energies of a $\Lambda-$particle in hypernuclei and are as follows[11]:

i) For the Gaussian potential:

$$D = 34.16 \ MeV, r_0 = 1.199 \ fm$$

ii) For the Poeschl-Teller type potential:

$$D = 38.9 \ MeV, r_0 = 0.986 \ fm$$

The results are displayed in the tables that follow by using the notation:

A_c:the mass number of the host nucleus

E_{hvt}:the total energy value obtained through the "HVT-HFT" scheme using the terms given in the table.

E_{an}:the total energy value from the analytic solution [Only s-states of the P.T. potential (45)]

E_p:the total energy value from a perturbation method [11c]. (Only for the $n = 0$ states of the P.T.-type potential)

E_{int}:the total energy value obtained through numerical integration.

$\tilde{V}$: $V_k \lambda^k < r^{2k+2} >$.

$< V >_{hvt}$:the expectation value for the potential energy through the (HVT-HFT) scheme.

$< V >_{an}$:the exact expectation value for the potential energy,given by Eq. (48). (Only s states of the P.T.-type potential)

$< V >_{int}$:the expectation value for the potential energy obtained through numerical integration.

$< r^2 >_{hvt}^{\frac{1}{2}}$:the root-mean-square radius of the particle orbit,obtained through the "HVT-HFT" scheme.

$< r^2 >_{int}^{\frac{1}{2}}$:The orbital radius,obtained through numerical integration.

The energy and $< r^2 >$ series obtained through the present scheme are provided on a term-by-tern basis, so that the accuracy ,for various hypernuclei, can be observed. Apparently,the first few terms of the HVT-series for the s-state energies (total and potential energies), in the whole range of A_c values studied, practically coincide with the ones extracted through numerical integration,or calculated by means of the corresponding exact expression in the case of the s-states of the P.T.-type potential (45).

For the lower excited states the accuracy of the HVT-energy values is in general quite satisfactory, in particular for the heavier elements. We should also keep in mind that the computation of various quantities through numerical integration is also subject to certain inaccuracies, which are expected, however, to be usually very small. We may also observe that the HVT-energy eigenvalues with n=0, for the P.T.-type potential (45), are also in quite good agreement with the values E_p(see Table IV), obtained with the perturbation method of Ref. 11c.The HVT values are in fact a little closer to those obtained by numerical integration, in comparison to the E_p ones.

.The reader can also observe that for the potential energy of the Poeschl-Teller type potential two tables are provided, each employing a different method of calculation for the $< V >$. Table V makes use of the statistical moments as they are given by Eq. (12).In fact the moments $< r^{2k+2} >$ are truncated at a such a term that the terms $\tilde{V}_k$ are all of order λ^5 in precision.

For example,we truncate $< r^2 >_{nl}$ so that the part used in Table V is

$$< r^2 >_{nl} \simeq C_2^{(0)} + C_2^{(1)}\lambda + C_2^{(2)}\lambda^2 + C_2^{(3)}\lambda^3 + C_2^{(4)}\lambda^4 + C_2^{(5)}\lambda^5 + ... \quad (59)$$

Therefore the term $\tilde{V}_0 = V_0\lambda^0 < r^2 >_{nl}$ is of order λ^5. By comparing Tables II and III one can see that the precision yielded by both methods is practically identical.What makes the difference is the simplicity of the second, which renders the first crude and time-consuming. In conclusion, the second method used in this paper, applied for the extraction of both $< T >$ and $< V >$,can be linked directly to all the previous papers, which have calculated the energy series for a great number of potentials. Thus one may obtain $< T >$ as well as $< V >$ by modifying the HVT-energy coefficients in a way analogous to that followed in the present paper. The remarks previously made regarding the accuracy of the HVT energy quantities, also hold, to some extent, for the orbital radii (Tables III and VII), although in that case the accuuracy is less satisfactory, mainly for certain states and in particular for the lighter elements.

We finally observe that the HVT values for both the energy quantities and the orbital radii are, for a given state, more accurate for the heavier elements than for the lighter ones. This is in accordance with our expectations since,as was observed in Sec. II, in the former case the parameter s^{-1} is smaller. Furthermore the accuracy of the results depends on the state (being more satisfactory for the lower states)

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TABLE I.THE TOTAL ENERGY FOR THE GAUSS POTENTIAL (see text)										
A_c	n	l	$E^{(0)}$	$E^{(1)}\lambda$	$E^{(2)}\lambda^2$	$E^{(3)}\lambda^3$	$E^{(4)}\lambda^4$	$E^{(5)}\lambda^5$	E_{hvt}	E_{int}
11	0	0	-5.24	-5.098	-0.299	-0.057	-0.0116	-0.0011	-10.71	-10.71
80	0	0	-19.87	-1.24	-0.036	-0.0039	-0.000341	-0.00034	-21.16	-21.16
80	0	1	-10.35	-2.9	-0.15	-0.027	-0.0065	-0.0034	-13.44	-13.45
80	0	2	-0.835	-5.22	-0.390	-0.10	-0.035	-0.019	-6.61	-6.61
80	1	0	-0.835	-6.21	-0.455	-0.10	-0.033	-0.021	-7.67	-7.67
80	1	1	8.68	-9.86	-0.94	-0.29	-0.11	-0.086	-2.62	-2.66
120	0	0	-21.71	-0.944	-0.023	-0.0019	-0.00017	-0.00004	-22.68	-22.68
120	0	1	-13.41	-2.20	-0.100	-0.015	-0.0033	-0.00074	-15.74	-15.74
120	0	2	-5.11	-3.96	-0.26	-0.058	-0.017	-0.006	-9.43	-9.43
120	0	3	3.17	-6.23	-0.53	-0.15	-0.059	-0.026	-3.83	-3.86
120	1	0	-5.11	-4.72	-0.30	-0.06	-0.016	-0.0047	-10.22	-10.23
120	1	1	3.17	-7.49	-0.622	-0.16	-0.060	-0.021	-5.19	-5.21

TABLE II.THE POTENTIAL ENERGY FOR THE GAUSS POTENTIAL (see text)										
A_c	n	l	$V^{(0)}$	$V^{(1)}\lambda$	$V^{(2)}\lambda^2$	$V^{(3)}\lambda^3$	$V^{(4)}\lambda^4$	$V^{(5)}\lambda^5$	$<V>_{hvt}$	$<V>_{int}$
11	0	0	14.44	0	0.149	0.057	0.017	0.0022	-19.47	-19.48
80	0	0	7.14	0	0.018	0.0033	0.00051	0.00003	-26.99	-26.99
80	0	1	11.90	0	-0.075	-0.027	-0.0098	-0.0036	-22.14	-22.13
80	0	2	16.66	0	0.197	0.015	0.0526	0.028	-17.11	-17.08
80	1	0	16.66	0	0.27	0.10	0.049	0.023	-17.09	-17.07
80	1	1	21.42	0	0.47	0.29	0.17	0.14	-11.68	-11.44
120	0	0	6.22	0	0.0119	0.0019	0.00025	0.00001	-27.92	-27.92
120	0	1	10.37	0	0.050	0.015	0.0049	0.0016	-23.71	-23.71
120	0	2	14.523	0	0.130	0.0058	0.0026	0.0012	-19.41	-19.40
120	0	3	18.66	0	0.268	0.153	0.090	0.055	-14.92	-14.81
120	1	0	14.52	0	0.15	0.061	0.0248	0.010	-19.39	-19.38
120	1	1	18.86	0	0.311	0.167	0.090	0.049	-14.87	-14.80

TABLE III.THE ORBITAL RADIUS FOR THE GAUSS POTENTIAL (see text)										
A_c	n	l	$C_2^{(0)}$	$C_2^{(1)}\lambda$	$C_2^{(2)}\lambda^2$	$C_2^{(3)}\lambda^3$	$C_2^{(4)}\lambda^4$	$C_2^{(5)}\lambda^5$	$\sqrt{<r^2>}_{hvt}$	$\sqrt{<r^2>}_{int}$
11	0	0	3.00	1.06	0.505	0.269	0.150	0.084	2.25	2.27
80	0	0	5.57	0.971	0.228	0.060	0.016	0.0045	2.62	2.62
80	0	1	9.29	2.26	0.77	0.304	0.149	0.054	3.58	3.58
80	0	2	13.01	4.08	1.81	0.933	0.654	0.305	4.56	4.6
80	1	0	13.01	4.85	2.31	1.209	0.667	0.381	4.73	4.79
80	1	1	16.73	7.71	4.58	3.01	2.24	1.51	5.98	6.38
120	0	0	6.37	0.967	0.198	0.045	0.010	0.0026	2.75	2.75
120	0	1	10.61	2.25	0.66	0.226	0.098	0.0314	3.72	3.72
120	0	2	14.86	4.06	1.57	0.705	0.431	0.175	4.67	4.68
120	0	3	19.11	6.38	3.05	1.70	1.40	0.66	5.68	5.78
120	1	0	14.86	4.83	2.00	0.913	0.439	0.218	4.82	4.84
120	1	1	19.11	7.67	3.97	2.28	1.477	0.870	5.94	6.08

TABLE IV.THE TOTAL ENERGY FOR THE P.T. POTENTIAL (45) (see text)												
A_c	n	l	$E^{(0)}-D$	$E^{(1)}\lambda$	$E^{(2)}\lambda^2$	$E^{(3)}\lambda^3$	$E^{(4)}\lambda^4$	$E^{(5)}\lambda^5$	E_{hvt}	E_{an}	E_p	E_{int}
6	0	0	8.84	-16.27	0.99	0	-0.01	0	-6.44	-6.44	-6.44	-6.44
9	0	0	1.64	-11.73	0.61	0	-0.004	0	-9.48	-9.48	-9.48	-9.48
11	0	0	1.64	-10.05	0.48	0	-0.003	0	-10.95	-10.95	-10.95	-10.95
15	0	0	-5.50	-7.96	0.34	0	-0.001	0	-13.13	-13.13	-13.13	-13.13
15	0	1	16.75	-18.57	0.26	-0.15	-0.081	-0.043	-1.84		-1.81	-1.91
31	0	0	-13.18	-4.72	0.15	0	0.0001	0	-17.75	-17.75	-17.75	-17.75
31	0	1	3.95	-11.01	0.14	-0.05	-0.02	-0.009	-7.02		-7.01	-7.03
80	0	0	-20.36	-2.45	0.05	0	0.00005	0	-22.76	-22.76	-22.76	-22.76
80	0	1	-8.01	-5.72	0.04	-0.01	-0.004	-0.001	-13.70		-13.70	-13.71
80	0	2	4.34	-10.30	-0.08	-0.07	-0.03	-0.013	-6.16		-6.14	-6.17
80	1	0	4.34	-12.26	0.13	0	0.0004	0	-7.78	-7.78		-7.78
80	1	1	16.69	-19.45	0.08	-0.049	-0.025	-0.013	-2.76			-2.78
120	0	0	-22.74	-1.86	0.038	0	-0.0004	0	-24.57	-24.57	-24.57	-24.57
120	0	1	-11.98	-4.34	0.03	-0.008	-0.002	-0.0005	-16.30			-16.31
120	0	2	-1.21	-7.82	-0.054	-0.045	-0.015	-0.005	-9.159		-9.14	-9.16
120	0	3	9.55	-12.29	-0.255	-0.143	-0.062	-0.03	-3.22		-3.19	-3.27
120	1	0	-1.21	-9.31	0.090	0	-0.0001	0	-10.43	-10.43		-10.44
120	1	1	9.55	-14.77	0.05	-0.028	-0.013	-0.005	-5.218			-5.22

TABLE V.THE POTENTIAL ENERGY FOR THE P.T.POTENTIAL (45) (see text)											
A_c	n	l	V_0	V_1	V_2	V_3	V_4	V_5	$<V>_{hvt}$	$<V>_{an}$	$<V>_{int}$
6	0	0	65	-117	166	-155	81	-17.9	-15.51	-15.51	-15.50
9	0	0	45.40	-60.90	73.86	-63.21	31.51	-6.74	-18.92	-18.92	-18.92
11	0	0	38.84	-45.44	50.64	-41.38	20.10	-4.23	-20.38	-20.37	-20.38
15	0	0	31.16	-29.82	29.06	-22.02	10.24	-2.10	-22.37	-22.37	-22.37
15	0	1	81.4	-131.84	155.75	-120.93	54.17	-10.52	-10.84		-10.57
31	0	0	20.134	-12.54	8.83	-5.49	2.28	-0.43	-26.12	-26.12	-26.12
31	0	1	44.73	-47.22	43.27	-29.05	11.93	-2.19	-17.42		-17.40
80	0	0	12.55	-4.79	2.20	-1.01	0.35	-0.06	-29.66	-29.66	-29.66
80	0	1	24.80	-15.40	9.63	-5.01	1.82	-0.30	-23.45		-23.47
80	0	2	41.88	-39.24	30.42	-17.52	6.32	-1.04	-17.08		-17.04
80	1	0	46.95	-50.84	44.40	-27.65	10.84	-1.78	-17.34	-17.34	-17.34
80	1	1	80.17	-120.14	124.33	-83.28	32.27	-5.47	-11.02		-10.97
120	0	0	10.47	-3.30	1.26	-0.51	0.16	-0.02	-30.84	-30.84	-30.84
120	0	1	20.03	-10.08	5.31	-2.50	0.83	-0.13	-25.44		-25.43
120	0	2	33.16	-24.34	16.16	-8.45	2.87	-0.45	-19.94		-19.93
120	0	3	52.14	-52.16	40.90	-22.87	7.89	-1.24	-14.24		-14.11
120	1	0	35.97	-31.34	23.49	-13.30	4.76	-0.78	-20.10	-20.10	-20.10
120	1	1	58.04	-70.26	63.80	-39.51	14.58	-2.39	-14.64		-14.62

TABLE VI.THE POTENTIAL ENERGY FOR THE P.T. POTENTIAL (45) (see text)											
A_c	n	l	$V^{(0)}$	$V^{(1)}\lambda$	$V^{(2)}\lambda^2$	$V^{(3)}\lambda^3$	$V^{(4)}\lambda^4$	$V^{(5)}\lambda^5$	$<V>_{hvt}$	$<V>_{an}$	$<V>_{int}$
11	0	0	18.75	0	-0.24	0	0.0046	0	-20.38	-20.37	-20.38
80	0	0	9.26	0	-0.029	0	0.00013	0	-29.66	-29.66	-29.66
80	0	1	15.44	0	-0.022	0.014	0.0065	0.0025	-23.45		-23.47
80	0	2	21.62	0	0.040	0.079	0.058	0.027	-17.07		-17.04
80	1	0	24.62	0	-0.068	0	0.0003	0	-17.34	-17.34	-17.34
80	1	1	27.79	0	-0.040	0.049	0.038	0.026	-11.02		-10.97
120	0	0	8.07	0	-0.019	0	0.00006	0	-30.84	-30.84	-30.84
120	0	1	13.48	0	-0.015	0.0084	0.0032	0.0011	-25.44		-25.43
120	0	2	18.84	0	0.027	0.045	0.028	0.0119	-19.94		-19.93
120	0	3	24.22	0	0.127	0.143	0.118	0.0632	-14.22		-14.11
120	1	0	18.84	0	-0.045	0	0.00016	0	-20.10	-20.10	-20.10
120	1	1	29.60	0	0.042	0.123	0.10	0.076	-14.64		-14.62

A_c	n	l	$C_2^{(0)}$	$C_2^{(1)}\lambda$	$C_2^{(2)}\lambda^2$	$C_2^{(3)}\lambda^3$	$C_2^{(4)}\lambda^4$	$C_2^{(5)}\lambda^5$	$\sqrt{<r^2>}_{hvt}$	$\sqrt{<r^2>}_{int}$
						TABLE VII.THE ORBITAL RADIUS FOR THE P.T.POTENTIAL (45) (see text)				
6	0	0	1.97	1.34	0.88	0.58	0.38	0.25	2.32	2.43
9	0	0	2.19	1.26	0.70	0.39	0.22	0.12	2.21	2.25
11	0	0	2.13	1.24	0.64	0.33	0.173	0.09	2.19	2.21
15	0	0	2.53	1.21	0.55	0.257	0.11	0.055	2.17	2.18
15	0	1	4.22	2.82	1.95	1.43	1.08	.84	3.51	4.04
31	0	0	3.17	1.16	0.41	0.14	0.05	0.01	2.22	2.22
31	0	1	5.28	2.71	1.45	0.81	0.47	0.28	3.32	3.39
80	0	0	4.29	1.13	0.29	0.07	0.01	0.004	2.41	2.41
80	0	1	7.16	2.65	1.02	0.414	0.174	0.075	3.39	3.39
80	0	2	10.03	4.77	2.46	1.35	0.78	0.47	4.46	4.55
80	1	0	10.03	5.69	3.06	1.63	0.87	0.47	4.67	4.72
80	1	1	12.89	9.02	6.14	4.19	2.90	2.03	6.10	6.53
120	0	0	4.90	1.13	0.25	0.056	0.012	0.002	2.52	2.52
120	0	1	8.18	2.64	0.88	0.31	0.11	0.04	3.49	3.49
120	0	2	11.45	4.75	2.13	1.02	0.52	0.27	4.49	4.52
120	0	3	14.72	7.47	4.19	2.55	1.64	1.10	5.63	5.87
120	1	0	11.45	5.66	2.66	1.23	0.57	*0.27	4.67	4.70
120	1	1	14.72	8.98	5.32	3.16	1.91	1.16 *	5.94	6.10