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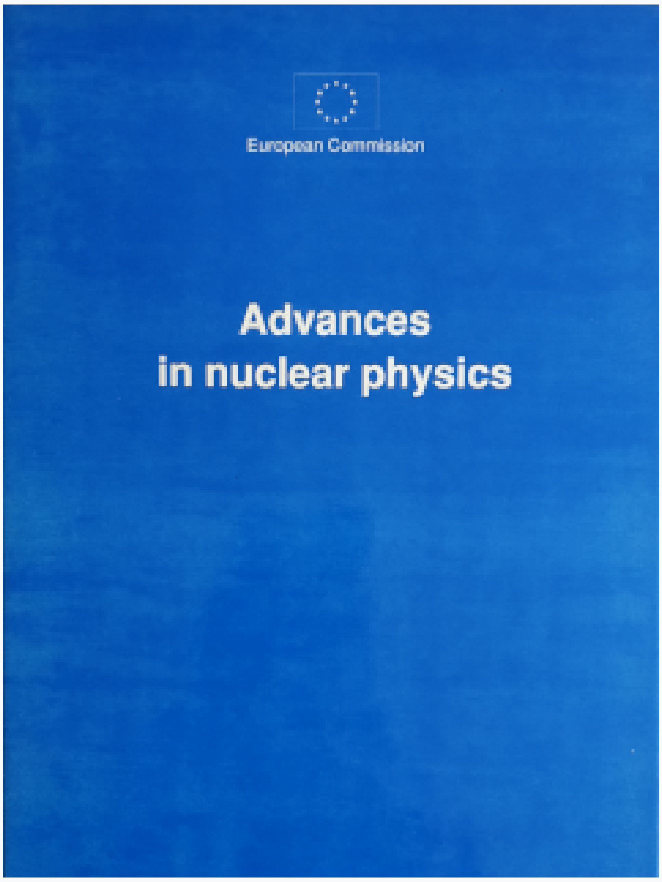

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Nonlinear extension of the u(3) algebra as the symmetry algebra of the three-dimensional anisotropic quantum harmonic oscillator with rational ratios of frequencies and the Nilsson model [1]

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Abstract

The symmetry algebra of the N-dimensional anisotropic quantum harmonic oscillator with rational ratios of frequencies is constructed by a method of general applicability to quantum superintegrable systems. The special case of the 3-dim oscillator is studied in more detail, because of its relevance in the description of superdeformed nuclei and nuclear and atomic clusters. In this case the symmetry algebra turns out to be a nonlinear extension of the u(3) algebra. A generalized angular momentum operator useful for labeling the degenerate states is constructed, clarifying the connection of the present formalism to the Nilsson model.

[1]Presented by C. Daskaloyannis

1. Introduction

Quantum algebras [1, 2] (also called quantum groups) are nonlinear generalizations of the usual Lie algebras, to which they reduce in the limiting case in which the deformation parameters are set equal to unity. From the mathematical point of view they have the structure of Hopf algebras [3]. The interest for applications of quantum algebras in physics was triggered in 1989 by the introduction of the q-deformed harmonic oscillator [4, 5, 6], although such mathematical structures have already been in existence [7, 8]. The q-deformed harmonic oscillator has been introduced as a tool for providing a boson realization for the quantum algebra $su_q(2)$. Since then several generalized deformed oscillators (see [9, 10, 11, 12, 13] and references therein), as well as generalized deformed versions of the su(2) algebra [14, 15, 16, 17, 18] have been introduced and used in several applications to physical problems.

On the other hand, the 3-dimensional anisotropic quantum harmonic oscillator is of interest in the description of many physical systems. Its single particle level spectrum is known [19, 20] to characterize the basic structure of superdeformed and hyperdeformed nuclei [21, 22]. Furthermore, it has been recently connected [23, 24] to the underlying geometrical structure in the Bloch–Brink α-cluster model [25]. It is also of interest for the interpretation of the observed shell structure in atomic clusters [26], especially after the realization that large deformations can occur in such systems [27]. The 2-dimensional anisotropic quantum harmonic oscillator is also of interest, since, for example, its single particle level spectrum characterizes the underlying symmetry of "pancake" nuclei [20].

The anisotropic harmonic oscillator in two [28, 29, 30, 31, 32, 33] and three [34, 35, 36, 37, 38, 39, 40] dimensions has been the subject of several investigations, both at the classical and at the quantum mechanical level. The special cases with frequency ratios 1:2 [41, 42] and 1:3 [43] have also been considered. The anisotropic harmonic oscillator is a well known example of a superintegrable system, both at the classical and at the quantum mechanical level [44, 45]. Although at the classical level it is clear that for the N-dimensional anisotropic harmonic oscillator the su(N) or sp(2N,R) algebras can be used for its description, the situation at the quantum level is not equally clear [28].

In section 2 of this letter the symmetry algebra of the N-dimensional anisotropic quantum harmonic oscillator with rational ratios of frequencies is studied by a method of general applicability to superintegrable systems [46], while in section 3 the case of the 3-dimensional oscillator is considered in more detail, because of its importance for the description of nuclei and possibly of atomic clusters. In the case of the 3-dim oscillator the symmetry algebra turns out to be a nonlinear generalization of u(3). In addition a generalized angular momentum operator is constructed, suitable for labelling the degenerate states in a way similar to that used in the framework of the

Nilsson model [47].

2. General case: the N-dimensional oscillator

Let us consider the system described by the Hamiltonian:

$$H = \frac{1}{2}\sum_{k=1}^{N}\left(p_k{}^2 + \frac{x_k^2}{m_k^2}\right), \tag{1}$$

where m_i are natural numbers mutually prime ones, i.e. their great common divisor is $\gcd(m_i, m_j) = 1$ for $i \neq j$ and $i, j = 1, \ldots, N$.

We define the creation and annihilation operators [28]:

$$\begin{aligned} a_k^\dagger &= \tfrac{1}{\sqrt{2}}\left(\tfrac{x_k}{m_k} - ip_k\right), \\ a_k &= \tfrac{1}{\sqrt{2}}\left(\tfrac{x_k}{m_k} + ip_k\right), \\ U_k &= \tfrac{1}{2}\left(p_k{}^2 + \tfrac{x_k^2}{m_k^2}\right) = \tfrac{1}{2}\left\{a_k, a_k^\dagger\right\}, \quad H = \sum_{k=1}^{N} U_k. \end{aligned} \tag{2}$$

These operators satisfy the relations (the indices k have been omitted):

$$\begin{aligned} a^\dagger U &= \left(U - \tfrac{1}{m}\right) a^\dagger \quad \text{or} \quad [U, \left(a^\dagger\right)^m] = \left(a^\dagger\right)^m, \\ a\,U &= \left(U + \tfrac{1}{m}\right) a \quad \text{or} \quad [U, (a)^m] = -\,(a)^m, \\ a^\dagger a &= U - \tfrac{1}{2m}, \qquad aa^\dagger = U + \tfrac{1}{2m}, \\ \left[a, a^\dagger\right] &= \tfrac{1}{m}, \\ \left(a^\dagger\right)^m (a)^m &= F(m, U), \\ (a)^m \left(a^\dagger\right)^m &= F(m, U + 1), \end{aligned} \tag{3}$$

where the function $F(m, x)$ is defined by:

$$F(m, x) = \prod_{p=1}^{m}\left(x - \frac{2p-1}{2m}\right). \tag{4}$$

Using the above relations we can define the enveloping algebra $\mathcal{C}$, defined by the polynomial combinations of the generators $\left\{1, H, \mathcal{A}_k, \mathcal{A}_k^\dagger, U_k\right\}$ and $k = 1, \ldots, N-1$, where:

$$\mathcal{A}_k^\dagger = \left(a_k^\dagger\right)^{m_k} (a_N)^{m_N}, \qquad \mathcal{A}_k = (a_k)^{m_k}\left(a_N^\dagger\right)^{m_N}. \tag{5}$$

These operators correspond to a multidimensional generalization of eq. (2):

$$[H, \mathcal{A}_k] = 0, \quad [H, \mathcal{A}_k^\dagger] = 0, \quad [H, U_k] = 0, \quad k = 1, \ldots, N-1. \tag{6}$$

The following relations are satisfied for $k \neq \ell$ and $k, \ell = 1, \ldots, N-1$:

$$[U_k, \mathcal{A}_\ell] = \left[U_k, \mathcal{A}_\ell^\dagger\right] = [\mathcal{A}_k, \mathcal{A}_\ell] = \left[\mathcal{A}_k^\dagger, \mathcal{A}_\ell^\dagger\right] = 0, \tag{7}$$

while

$$\begin{aligned}
U_k \mathcal{A}_k^\dagger &= \mathcal{A}_k^\dagger \left(U_k + 1\right), \\
U_k \mathcal{A}_k &= \mathcal{A}_k \left(U_k - 1\right), \\
\mathcal{A}_k^\dagger \mathcal{A}_k &= F\Big(m_k, U_k\Big) F\Big(m_N, H - \sum_{\ell=1}^{N-1} U_\ell + 1\Big), \\
\mathcal{A}_k \mathcal{A}_k^\dagger &= F\Big(m_k, U_k + 1\Big) F\Big(m_N, H - \sum_{\ell=1}^{N-1} U_\ell\Big).
\end{aligned} \tag{8}$$

One additional relation for $k \neq \ell$ can be derived:

$$F\Big(m_N, H - \sum_{\ell=1}^{N-1} U_\ell\Big) \mathcal{A}_k^\dagger \mathcal{A}_\ell = F\Big(m_N, H - \sum_{\ell=1}^{N-1} U_\ell + 1\Big) \mathcal{A}_\ell \mathcal{A}_k^\dagger. \tag{9}$$

The algebra defined by the above relations accepts a Fock space representation. The elements of the basis $|E, p_1, \ldots, p_{N-1}\rangle$ are characterized by the eigenvalues of the N commuting elements of the algebra H and U_k with $k = 1, \ldots, N-1$. The elements $\mathcal{A}_k$ and $\mathcal{A}_k^\dagger$ are the corresponding ladder operators of the algebra. The following relations hold:

$$\begin{aligned}
&H\,|E, p_1, \ldots, p_{N-1}\rangle = E\,|E, p_1, \ldots, p_{N-1}\rangle, \\
&U_k\,|E, p_1, \ldots, p_{N-1}\rangle = p_k\,|E, p_1, \ldots, p_{N-1}\rangle, \\
&\mathcal{A}_k\,|E, p_1, \ldots, p_k, \ldots, p_{N-1}\rangle = \\
&\sqrt{F\Big(m_k, p_k\Big) F\Big(m_N, E - \sum_{\ell=1}^{N-1} p_\ell + 1\Big)}\,|E, p_1, \ldots, p_k - 1, \ldots, p_{N-1}\rangle, \\
&\mathcal{A}_k^\dagger\,|E, p_1, \ldots, p_k, \ldots, p_{N-1}\rangle = \\
&\sqrt{F\Big(m_k, p_k + 1\Big) F\Big(m_N, E - \sum_{\ell=1}^{N-1} p_\ell\Big)}\,|E, p_1, \ldots, p_k + 1, \ldots, p_{N-1}\rangle.
\end{aligned} \tag{10}$$

Let $p_k^{\min}$ be the minimum value of p_k such that

$$\mathcal{A}_k \Big|E, p_1, \ldots, p_k^{\min}, \ldots, p_{N-1}\Big\rangle = 0. \tag{11}$$

From eq. (10) we find that we must have:

$$F\Big(m_k, p_k^{\min}\Big) = 0. \tag{12}$$

Then p_k is one of the roots of the function F defined by eq. (4). The general form of the roots is:

$$p_k^{\min} = \frac{2q_k - 1}{2m_k}, \qquad q_k = 1, \ldots, m_k. \tag{13}$$

Each root is characterized by a number q_k. The numbers q_k also characterize the representations of the algebra, as we shall see.

The elements of the Fock space can be generated by successive applications of the ladder operators $\mathcal{A}_k^\dagger$ on the minimum weight element

$$\left|E, p_1^{\min}, \ldots, p_{N-1}^{\min}\right\rangle = \left|\begin{array}{c} E, \{0\} \\ {[q]} \end{array}\right\rangle = \left|\begin{array}{c} E, 0, \ldots, 0 \\ q_1, \ldots, q_k, \ldots, q_N \end{array}\right\rangle. \tag{14}$$

The elements of the basis of the Fock space are given by:

$$\left|E, [p_k^{\min} + n_k]\right\rangle = \left|\begin{array}{c} E, \{n\} \\ {[q]} \end{array}\right\rangle = \frac{1}{\sqrt{C_{[q]}^{\{n\}}}} \left(\prod_{k=1}^{N-1} \left(\mathcal{A}_k^\dagger\right)^{n_k}\right) \left|\begin{array}{c} E, \{0\} \\ {[q]} \end{array}\right\rangle, \tag{15}$$

where $\{n\} = (n_1, n_2, \cdots, n_{N-1})$ and $[q] = (q_1, q_2, \ldots, q_N)$, while $C_{[q]}^{\{n\}}$ are normalization coefficients.

The generators of the algebra acting on the base of the Fock space give:

$$\begin{aligned} H\left|\begin{array}{c} E, \{n\} \\ {[q]} \end{array}\right\rangle &= E\left|\begin{array}{c} E, \{n\} \\ {[q]} \end{array}\right\rangle, \\ U_k\left|\begin{array}{c} E, \{n\} \\ {[q]} \end{array}\right\rangle &= \left(n_k + p_k^{\min}\right)\left|\begin{array}{c} E, \{n\} \\ {[q]} \end{array}\right\rangle, \quad k = 1, \ldots, N-1, \end{aligned} \tag{16}$$

$$\begin{aligned} &\mathcal{A}_k^\dagger\left|\begin{array}{c} E, \{n\} \\ {[q]} \end{array}\right\rangle = \\ &= \sqrt{F\left(m_k, n_k + p_k^{\min} + 1\right) F\left(m_N, E - \sum_{\ell=1}^{N-1} \left(n_\ell + p_\ell^{\min}\right)\right)} \cdot \\ &\cdot\left|\begin{array}{c} E, n_1, \ldots, n_k + 1, \ldots, n_{N-1} \\ q_1, \ldots, q_k, \ldots, q_N \end{array}\right\rangle, \end{aligned} \tag{17}$$

$$\begin{aligned} &\mathcal{A}_k\left|\begin{array}{c} E, \{n\} \\ {[q]} \end{array}\right\rangle = \\ &= \sqrt{F\left(m_k, n_k + p_k^{\min}\right) F\left(m_N, E - \sum_{\ell=1}^{N-1} \left(n_\ell + p_\ell^{\min}\right) + 1\right)} \cdot \\ &\cdot\left|\begin{array}{c} E, n_1, \ldots, n_k - 1, \ldots, n_{N-1} \\ q_1, \ldots, q_k, \ldots, q_N \end{array}\right\rangle. \end{aligned} \tag{18}$$

The existence of a finite dimensional representation implies that after Σ successive applications of the ladder operators $\mathcal{A}^\dagger$ on the minimum weight element one gets zero, so that the following condition is satisfied:

$$F(m_N, E - \Sigma - \sum_{\ell=1}^{N-1} p_\ell^{\min}) = 0.$$

Therefore:

$$E - \Sigma - \sum_{\ell=1}^{N-1} p_\ell^{\min} = p_N^{\min},$$

where $p_N^{\min}$ is the root of equation $F(m_N, p_N^{\min}) = 0$. Then

$$E = \Sigma + \sum_{k=1}^{N} \frac{2q_k - 1}{2m_k}. \tag{19}$$

In the case of finite dimensional representations only the energies given by eq. (19) are permitted and the elements of the Fock space can be described by using Σ instead of E. The action of the generators on the Fock space is described by the following relations:

$$\begin{aligned} H\left| \begin{matrix} \Sigma, \{n\} \\ [q] \end{matrix} \right\rangle &= \left(\Sigma + \sum_{k=1}^{N} \tfrac{2q_k-1}{2m_k}\right)\left| \begin{matrix} \Sigma, \{n\} \\ [q] \end{matrix} \right\rangle, \\ U_k\left| \begin{matrix} \Sigma, \{n\} \\ [q] \end{matrix} \right\rangle &= \left(n_k + \tfrac{2q_k-1}{2m_k}\right)\left| \begin{matrix} \Sigma, \{n\} \\ [q] \end{matrix} \right\rangle, \quad k = 1, \ldots, N-1, \end{aligned} \tag{20}$$

$$\begin{aligned} &\mathcal{A}_k^\dagger\left| \begin{matrix} \Sigma, \{n\} \\ [q] \end{matrix} \right\rangle = \\ &= \sqrt{F\left(m_k, n_k + \tfrac{2q_k-1}{2m_k} + 1\right) F\left(m_N, \Sigma - \sum_{\ell=1}^{N-1} n_\ell + \tfrac{2q_N-1}{2m_N}\right)} \cdot \\ &\cdot \left| \begin{matrix} \Sigma, n_1, \ldots, n_k + 1, \ldots, n_{N-1} \\ q_1, \ldots, q_k, \ldots, q_N \end{matrix} \right\rangle, \end{aligned} \tag{21}$$

$$\begin{aligned} &\mathcal{A}_k\left| \begin{matrix} \Sigma, \{n\} \\ [q] \end{matrix} \right\rangle = \\ &= \sqrt{F\left(m_k, n_k + \tfrac{2q_k-1}{2m_k}\right) F\left(m_N, \Sigma - \sum_{\ell=1}^{N-1} n_\ell + \tfrac{2q_N-1}{2m_N} + 1\right)} \cdot \\ &\cdot \left| \begin{matrix} \Sigma, n_1, \ldots, n_k - 1, \ldots, n_{N-1} \\ q_1, \ldots, q_k, \ldots, q_N \end{matrix} \right\rangle. \end{aligned} \tag{22}$$

The dimension of the representation is given by

$$d = \binom{\Sigma + N - 1}{\Sigma} = \frac{(\Sigma+1)(\Sigma+2)\cdots(\Sigma+N-1)}{(N-1)!}.$$

It is clear that to each value of Σ correspond $m_1 m_2 \ldots m_N$ energy eigenvalues, each eigenvalue having degeneracy d.

Using eqs (3) we can prove that the algebra generated by the generators $a_\ell^\dagger, a_\ell, N_\ell = m_\ell U_\ell - 1/2$ is an oscillator algebra with structure function [9, 12]

$$\Phi_\ell(x) = x/m_\ell,$$

i.e. with

$$\Phi_l(N_l) = U_l - \frac{1}{2m_l}.$$

This oscillator algebra is characterized by the commutation relations:

$$\left[N_\ell, a_\ell^\dagger\right] = a_\ell^\dagger, \quad [N_\ell, a_\ell] = a_\ell^\dagger, \quad a_\ell^\dagger a_\ell = \Phi_\ell(N_\ell), \quad a_\ell a_\ell^\dagger = \Phi_\ell(N_\ell + 1). \quad (23)$$

There are in total N different oscillators of this type, uncoupled to each other. The Fock space corresponding to these oscillators defines an infinite dimensional representation of the algebra defined eqs (6-9). In order to see the connection of the present basis to the usual Cartesian basis, one can use for the latter the symbol $[r] = (r_1, r_2, \ldots, r_N)$. One then has

$$\begin{aligned} a_\ell^\dagger \,|[r]\rangle &= \sqrt{\Phi(r_\ell + 1)}\, |r_1, \ldots, r_\ell + 1, \ldots, r_N\rangle, \\ a_\ell \,|[r]\rangle &= \sqrt{\Phi(r_\ell)}\, |r_1, \ldots, r_\ell - 1, \ldots, r_N\rangle, \\ N_\ell \,|[r]\rangle &= r_\ell \,|[r]\rangle. \end{aligned}$$

The connection between the above basis and the basis defined by eqs (20-22) is given by:

$$|[r]\rangle = \left| \begin{matrix} \Sigma, \{n\} \\ [q] \end{matrix} \right\rangle,$$

$$\begin{matrix} r_\ell = n_\ell m_\ell + \mathrm{mod}\,(r_\ell, m_\ell) \\ \ell = 1, \ldots, N \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} n_k = [r_k/m_k] \\ k = 1, \ldots, N-1 \\ q_\ell = \mathrm{mod}\ (r_\ell, m_\ell) + 1 \\ \Sigma = \sum\limits_{\ell=1}^{N} [r_\ell/m_\ell] \end{matrix} \qquad (24)$$

where $[x]$ means the integer part of the number x.

Using the correspondence between the present basis and the usual Cartesian basis, given in eq. (24), the action of the operators $a_k^\dagger$ on the present basis can be calculated for $k = 1, \ldots, N-1$:

$$a_k^\dagger \left| \begin{matrix} \Sigma, \{n\} \\ [q] \end{matrix} \right\rangle = \sqrt{n_k + q_k/m_k} \left| \begin{matrix} \Sigma', \{n'\} \\ [q'] \end{matrix} \right\rangle,$$

where

$$\begin{aligned} n'_\ell = n_\ell \quad q'_\ell &= q_\ell \quad \text{for } \ell \neq k, \\ n'_k &= n_k + [q_k/m_k], \\ \Sigma' &= \Sigma + [q_k/m_k], \\ q'_k &= \mathrm{mod}\ (q_k, m_k) + 1, \end{aligned}$$

while for the operator $a_N^\dagger$ one has

$$a_N^\dagger \left| \begin{matrix} \Sigma, \{n\} \\ [q] \end{matrix} \right\rangle = \sqrt{\Sigma - \sum_{k=1}^{N-1} n_k + q_N/m_N} \left| \begin{matrix} \Sigma', \{n'\} \\ [q'] \end{matrix} \right\rangle,$$

where

$$n'_k = n_k, \qquad q'_k = q_k, \qquad \text{for } k = 1, \ldots, N-1,$$
$$\Sigma' = \Sigma + [q_N/m_N],$$
$$q'_N = \text{mod } (q_N, m_N) + 1.$$

Similarly for the operators a_k one can find for $k = 1, \ldots, N-1$:

$$a_k \left| \begin{array}{c} \Sigma, \{n\} \\ {[q]} \end{array} \right\rangle = \sqrt{n_k + (q_k - 1)/m_k} \left| \begin{array}{c} \Sigma', \{n'\} \\ {[q']} \end{array} \right\rangle,$$

where

$$n'_\ell = n_\ell, \qquad q'_\ell = q_\ell, \qquad \text{for } \ell \neq k,$$
$$n'_k = n_k + [(q_k - 2)/m_k],$$
$$\Sigma' = \Sigma + [(q_k - 2)/m_k],$$
$$q'_k = \text{mod } (q_k - 2, m_k) + 1,$$

while for the operator a_N one has

$$a_N \left| \begin{array}{c} \Sigma, \{n\} \\ {[q]} \end{array} \right\rangle = \sqrt{\Sigma - \sum_{k=1}^{N-1} n_k + (q_N - 1)/m_N} \left| \begin{array}{c} \Sigma', \{n'\} \\ {[q']} \end{array} \right\rangle,$$

where

$$n'_k = n_k, \qquad q'_k = q_k, \qquad \text{for } k = 1, \ldots, N-1,$$
$$\Sigma' = \Sigma + [(q_N - 2)/m_N],$$
$$q'_N = \text{mod } (q_N - 2, m_N) + 1.$$

An important difference between the operators a_k, $a_k^\dagger$ used here and the operators $A_i^{(s)}$, $A_i^{(s)\dagger}$ used in refs [34, 36, 37, 40] has to be pointed out. The operators used here satisfy the relations

$$[a_k, a_l^\dagger] = \frac{1}{m_k} \delta_{kl},$$

i.e. they represent oscillators completely decoupled from each other, while the operators of refs [34, 36, 37, 40] satisfy the relations

$$[A_i^{(s)}, A_j^{(s)\dagger}] = \delta_{ij}, \qquad [A_i^{(s)}, A_j^{(s')\dagger}] \neq \delta_{ij},$$

i.e. they represent oscillators not completely decoupled. Notice that (s) of refs [34, 36, 37] ($\{\lambda\}$ of ref. [40]) is analogous to the $[q]$ used in the present work.

3. The 3-dimensional oscillator and relation to the Nilsson model

In this section the 3-dim case will be studied in more detail, because of its relevance for the description of superdeformed nuclei and of nuclear and atomic clusters. The 3-dim anisotropic oscillator is the basic ingredient of the Nilsson model [47], which in addition contains a term proportional to the square of the angular momentum operator, as well as a spin-orbit coupling term, the relevant Hamiltonian being

$$H_{Nilsson} = H_{osc} - 2k\vec{L}\cdot\vec{S} - k\nu\vec{L}^2,$$

where k, ν are constants. The spin-orbit term is not needed in the case of atomic clusters, while in the case of nuclei it can be effectively removed through a unitary transformation, both in the case of the spherical Nilsson model [48, 49, 50] and of the axially symmetric one [51, 52]. An alternative way to effectively remove the spin-orbit term in the spherical Nilsson model is the q-deformation of the relevant algebra [53]. It should also be noticed that the spherical Nilsson Hamiltonian is known to possess an osp(1|2) supersymmetry [54].

In the case of the 3-dim oscillator the relevant operators of eq. (2) form a nonlinear generalization of the algebra u(3), the q-deformed version of which can be found in [55, 56, 57].

As we have already seen, to each Σ value correspond $m_1m_2m_3$ energy eigenvalues, each eigenvalue having degeneracy $(\Sigma+1)(\Sigma+2)/2$. In order to distinguish the degenerate eigenvalues, we are going to introduce some generalized angular momentum operators, L_i $(i=1,2,3)$, defined by:

$$L_k = i\epsilon_{ijk}\left((a_i)^{m_i}\left(a_j^\dagger\right)^{m_j} - \left(a_i^\dagger\right)^{m_i}(a_j)^{m_j}\right). \tag{25}$$

One can prove that

$$L_1 = i\left(\mathcal{A}_2 - \mathcal{A}_2^\dagger\right), \quad L_2 = i\left(\mathcal{A}_1^\dagger - \mathcal{A}_1\right).$$

The following commutation relations can be verified:

$$[L_i, L_j] = i\epsilon_{ijk}\left(F(m_k, U_k+1) - F(m_k, U_k)\right)L_k. \tag{26}$$

It is worth noticing that in the case of $m_1 = m_2 = m_3 = 1$ one has $F(1,x) = x - 1/2$, so that the above equation gives the usual angular momentum commutation relations.

The operators defined in eq. (25) commute with the oscillator hamiltonian H and therefore conserve the number Σ which characterizes the dimension of the representation. Also these operators do not change the numbers q_1, q_2, q_3 as we can see from eqs (21-22). The eigenvalues of these operators can be calculated using Hermite function techniques.

Let us consider in particular the generalized angular momentum projection:

$$L_3 = i\left((a_1)^{m_1}\left(a_2^\dagger\right)^{m_2} - \left(a_1^\dagger\right)^{m_1}(a_2)^{m_2}\right). \tag{27}$$

This acts on the basis as follows

$$L_3 \left| \begin{array}{c} \Sigma, \{n\} \\ [q] \end{array} \right\rangle =$$

$$= i\Big(\sqrt{F\left(m_1, n_1 + \tfrac{2q_1-1}{2m_1}\right) F\left(m_2, n_2 + \tfrac{2q_2-1}{2m_2} + 1\right)} \left| \begin{array}{c} \Sigma, n_1 - 1, n_2 + 1 \\ [q] \end{array} \right\rangle -$$

$$- \sqrt{F\left(m_1, n_1 + \tfrac{2q_1-1}{2m_1} + 1\right) F\left(m_2, n_2 + \tfrac{2q_2-1}{2m_2}\right)} \left| \begin{array}{c} \Sigma, n_1 + 1, n_2 - 1 \\ [q] \end{array} \right\rangle\Big). \tag{28}$$

This operator conserves the quantum number

$$j = \frac{n_1 + n_2}{2}, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

In addition one can introduce the quantum number

$$m = \frac{n_1 - n_2}{2}.$$

One can then replace the quantum numbers n_1, n_2 by the quantum numbers j, μ, where μ is the eigenvalue of the L_3 operator. The new representation basis one can label as

$$L_3 \left| \begin{array}{c} \Sigma, \\ j, \mu \\ [q] \end{array} \right\rangle = \mu \left| \begin{array}{c} \Sigma, \\ j, \mu \\ [q] \end{array} \right\rangle. \tag{29}$$

This basis is connected to the basis of the previous section as follows

$$\left| \begin{array}{c} \Sigma, \\ j, \mu \\ [q] \end{array} \right\rangle = \sum_{m=-j}^{j} \frac{c[j, m, \mu]}{\sqrt{[j+m]_1! [j-m]_2!}} \left| \begin{array}{c} \Sigma, j+m, j-m \\ [q] \end{array} \right\rangle, \tag{30}$$

where

$$[0]_k! = 1, \quad [n]_k! = [n]_k [n-1]_k!, \quad [n]_k = F\left(m_k, n + \frac{2q_k - 1}{2m_k}\right),$$

and the coefficients $c[j, m, \mu]$ in eq. (30) satisfy the recurrence relation:

$$\mu c[j, m, \mu] = i\left([j-m]_2 c[j, m+1, \mu] - [j+m]_1 c[j, m-1, \mu]\right). \tag{31}$$

These relations can be satisfied only for special values of the parameter μ, corresponding to the eigenvalues of L_3. It is worth noticing that in the case of $m_1 = m_2$, which corresponds to axially symmetric oscillators, the possible values turn out to be $\mu = -2j, -2(j-1), \ldots, 2(j-1), 2j$. In nuclear physics the quantum numbers $n_\perp = n_1 + n_2$ and $\Lambda = \pm n_\perp$, $\pm(n_\perp - 2)$, ..., ± 1 or 0 are used [58]. From the above definitions it is clear that $j = n_\perp/2$ and

$\mu = \Lambda$. Therefore in the case of $m_1 = m_2$, which includes axially symmetric prolate nuclei with $m_1 : m_2 : m_3 = 1 : 1 : m$, as well as axially symmetric oblate nuclei with $m_1 : m_2 : m_3 = m : m : 1$, the correspondence between the present scheme and the Nilsson model is clear.

In the case of axially symmetric prolate nuclei ($m_1 = m_2 = 1$) one can easily check that the expression

$$L^2 = L_1^2 + L_2^2 + (F(m_3, U_3 + 1) - F(m_3, U_3))L_3^2$$

satisfies the commutation relations

$$[L^2, L_i] = 0, \qquad i = 1, 2, 3.$$

Thus L^2 is the second order Casimir operator of the algebra closed by L_1, L_2, L_3.

4. Discussion

The symmetry algebra of the N-dim anisotropic quantum harmonic oscillator with rational ratios of frequencies has been constructed by a method [46] of general applicability in constructing finite-dimensional representations of quantum superinegrable systems. The case of the 3-dim oscillator has been considered in detail, because of its relevance to the single particle level spectrum of superdeformed and hyperdeformed nuclei [19, 20], to the underlying geometrical structure of the Bloch-Brink α-cluster model [23, 24], and possibly to the shell structure of atomic clusters at large deformations [26, 27]. The symmetry algebra in this case is a nonlinear generalization of the u(3) algebra. For labeling the degenerate states, generalized angular momentum operators are introduced, clarifying the connection of the present approach to the Nilsson model.

In the case of the 2-dim oscillator with ratio of frequencies $2:1$ ($m_1 = 1$, $m_2 = 2$) it has been shown [59] that the relevant nonlinear generalized u(2) algebra can be identified as the finite W algebra $\mathrm{W}_3^{(2)}$ [60, 61]. In the case of the 3-dim axially symmetric

oblate oscillator with frequency ratio $1:2$ (which corresponds to the case $m_1 = m_2 = 2$, $m_3 = 1$) the relevant symmetry is related to O(4) [40, 62]. The search for further symmetries, related to specific frequency ratios, hidden in the general nonlinear algebraic framework given in this work is an interesting problem.

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