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Symmetry algebra of the planar anisotropic quantum harmonic oscillator with rational ratio of frequencies [1]

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Abstract

The symmetry algebra of the two-dimensional quantum harmonic oscillator with rational ratio of frequencies is identified as a non-linear extension of the u(2) algebra. The finite dimensional representation modules of this algebra are studied and the energy eigenvalues are determined using algebraic methods of general applicability to quantum superintegrable systems.

The two-dimensional anisotropic harmonic oscillator with rational ratio of frequencies is a well known example of a classical superintegrable system [1]. It has a third integral of motion which is given implicitly as a function of the energy [2], or as a polynomial function of the coordinates and the momenta [3]. This oscillator is the natural generalization of the isotropic harmonic oscillator, both of them widely used in many branches of physics. The cases with ratio of frequencies 1/2 and 1/3 have been studied in [4] and [5], respectively.

The quantum mechanical analogues of the isotropic and anisotropic oscillators are well known completely solvable models. The symmetry algebra of

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[1]Presented by C. Daskaloyannis

the isotropic quantum harmonic oscillator in a space with constant curvature, as well as the symmetry algebra of the corresponding Kepler problem, have been identified in [6,7]. From the algebraic point of view, however, while the many-dimensional analogue of the isotropic quantum oscillator can be studied using the su(N) or sp(2N,R) algebras, even for the two-dimensional anisotropic quantum oscillator the symmetry algebra is still missing. Since 1940 Jauch and Hill [3] have pointed out that "the two-dimensional anisotropic oscillator has the same symmetry as the isotropic oscillator in classical mechanics, but the quantum mechanical problem presents complications which leave its symmetry group in doubt". The case of ratio of frequencies equal to 1/2 has been studied in [8], using the fact that the Hamiltonian is separable in parabolic coordinates. The algebra describing the symmetry is a quadratic Lie algebra [9]. The importance of the quadratic extensions of Lie algebras in the study of symmetries was pointed out by several authors [10,11]. In the case of ratio of frequencies equal to 1/3 the Hamiltonian is not separable, but the problem can be solved [12], the symmetry being described by a cubic Lie algebra. About the general problem of rational ratio of frequencies Jauch and Hill [3] noticed that "the difficulties encountered in this relatively simple problem throw question on the true interpretation of classical multiply-periodic motions in quantum mechanics".

In this letter we identify the symmetry algebra of the two-dimensional anisotropic quantum harmonic oscillator, when the ratio of frequencies is a rational number. The techniques used are related to the q-deformed oscillator [13,14,15], introduced as a tool appropriate for constructing boson realizations of quantum algebras (quantum groups) [16], which are non-linear generalizations of the usual Lie algebras. Several mutually equivalent generalizations of the concept of the deformed oscillator already exist [17], their connection to N=2 supersymmetric quantum mechanics being also discussed [18]. The formalism to be used here is the one of [19]. The usefulness of non-linear generalizations of Lie algebras in the study of integrable systems has been demonstrated in the case of the Calogero model [20,21].

Let us consider the system described by the Hamiltonian:

$$H = \frac{1}{2}\left(p_x{}^2 + p_y{}^2 + \frac{x^2}{m^2} + \frac{y^2}{n^2}\right), \tag{1}$$

where m and n are two natural numbers mutually prime ones, i.e. their great common divisor is $\gcd(m,n) = 1$.

We define the creation and annihilation operators:

$$a^\dagger = \frac{x/m - ip_x}{\sqrt{2}}, \quad a = \frac{x/m + ip_x}{\sqrt{2}},$$

$$b^\dagger = \frac{y/n - ip_y}{\sqrt{2}}, \quad b = \frac{y/n + ip_y}{\sqrt{2}}.$$

(2)

These operators satisfy the commutation relations:

$$\left[a, a^\dagger\right] = \frac{1}{m}, \quad \left[b, b^\dagger\right] = \frac{1}{n}, \quad \text{other commutators} = 0.$$

(3)

Using eqs (2) and (3) we can prove by induction that:

$$\left[a, \left(a^\dagger\right)^p\right] = \frac{p}{m}\left(a^\dagger\right)^{p-1}, \quad \left[b, \left(b^\dagger\right)^p\right] = \frac{p}{n}\left(b^\dagger\right)^{p-1},$$

$$\left[a^\dagger, (a)^p\right] = -\frac{p}{m}(a)^{p-1}, \quad \left[b^\dagger, (b)^p\right] = -\frac{p}{n}(b)^{p-1}.$$

Defining

$$U = \frac{1}{2}\left\{a, a^\dagger\right\}, \quad W = \frac{1}{2}\left\{b, b^\dagger\right\},$$

one can easily prove that:

$$\left[U, \left(a^\dagger\right)^p\right] = \frac{p}{m}\left(a^\dagger\right)^p, \quad \left[W, \left(b^\dagger\right)^p\right] = \frac{p}{n}\left(b^\dagger\right)^p,$$

$$[U, (a)^p] = -\frac{p}{m}(a)^p, \quad [W, (b)^p] = -\frac{p}{n}(b)^p.$$

Using the above properties we can define the enveloping algebra generated by the operators:

$$S_+ = \left(a^\dagger\right)^m (b)^n, \quad S_- = (a)^m \left(b^\dagger\right)^n,$$

$$S_0 = \tfrac{1}{2}(U - W), \quad H = U + W.$$

(4)

These genarators satisfy the following relations:

$$[S_0, S_\pm] = \pm S_\pm, \quad [H, S_i] = 0, \quad \text{for} \quad i = 0, \pm,$$

(5)

and

$$S_+ S_- = \prod_{k=1}^{m}\left(U - \frac{2k-1}{2m}\right)\prod_{\ell=1}^{n}\left(W + \frac{2\ell-1}{2n}\right),$$

$$S_- S_+ = \prod_{k=1}^{m} \left(U + \frac{2k-1}{2m} \right) \prod_{\ell=1}^{n} \left(W - \frac{2\ell-1}{2n} \right).$$

The above relations mean that the harmonic oscillator of eq. (1) is described by the enveloping algebra of the non-linear generalization of the U(2) algebra formed by the generators S_0, S_+, S_- and H, satisfying the commutation relations of eq. (5) and

$$[S_-, S_+] = F_{m,n}(H, S_0 + 1) - F_{m,n}(H, S_0),$$

(6)

where $\quad F_{m,n}(H, S_0) = \prod_{k=1}^{m} \left(H/2 + S_0 - \frac{2k-1}{2m} \right) \prod_{\ell=1}^{n} \left(H/2 - S_0 + \frac{2\ell-1}{2n} \right).$

This algebra is a non-linear generalization of the U(2) algebra, of order $m + n - 1$. In the case of $m/n = 1/1$ this algebra is the usual U(2) algebra, and the operators $S_0, S_\pm$ satisfy the commutation relations of the ordinary SU(2) algebra.

The finite dimensional representation modules of this algebra can be found using the deformed oscillator algebra [19]. The operators:

$$\mathcal{A}^\dagger = S_+, \quad \mathcal{A} = S_-, \quad \mathcal{N} = S_0 - u, \quad u = \text{ constant}, \quad (7)$$

where u is a constant to be determined, are the generators of a deformed oscillator algebra:

$$\left[\mathcal{N}, \mathcal{A}^\dagger \right] = \mathcal{A}^\dagger, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \quad \mathcal{A}^\dagger \mathcal{A} = \Phi(H, \mathcal{N}), \quad \mathcal{A} \mathcal{A}^\dagger = \Phi(H, \mathcal{N} + 1).$$

The structure function Φ of this algebra is determined by the function $F_{m,n}$ in eq. (6):

$$\Phi(H, \mathcal{N}) = F_{m,n}(H, \mathcal{N} + u) =$$
$$= \prod_{k=1}^{m} \left(H/2 + \mathcal{N} + u - \frac{2k-1}{2m} \right) \prod_{\ell=1}^{n} \left(H/2 - \mathcal{N} - u + \frac{2\ell-1}{2n} \right). \quad (8)$$

The deformed oscillator corresponding to the structure function of eq. (8) has an energy dependent Fock space of dimension $N+1$ if

$$\Phi(E, 0) = 0, \quad \Phi(E, N+1) = 0, \quad \Phi(E, k) > 0, \quad \text{for} \quad k = 1, 2, \ldots, N. \quad (9)$$

The Fock space is defined by:

$$H|E, k> = E|E, k>, \quad \mathcal{N}|E, k> = k|E, k>, \quad a|E, 0> = 0, \quad (10)$$

$$\mathcal{A}^\dagger |E, k> = \sqrt{\Phi(E, k+1)}|E, k+1>, \quad \mathcal{A}|E, k> = \sqrt{\Phi(E, k)}|E, k-1>.$$
(11)

The basis of the Fock space is given by:

$$|E, k> = \frac{1}{\sqrt{[k]!}} \left(\mathcal{A}^\dagger\right)^k |E, 0>, \quad k = 0, 1, \ldots N,$$

where the "factorial" $[k]!$ is defined by the recurrence relation:

$$[0]! = 1, \quad [k]! = \Phi(E, k)[k-1]! \quad .$$

Using the Fock basis we can find the matrix representation of the deformed oscillator and then the matrix representation of the algebra of eqs. (5), (6). The solution of eqs (9) implies the following pairs of permitted values for the energy eigenvalue E and the constant u:

$$E = N + \frac{2p-1}{2m} + \frac{2q-1}{2n},$$
(12)

where $p = 1, 2, \ldots, m$, $q = 1, 2, \ldots, n$, and

$$u = \frac{1}{2}\left(\frac{2p-1}{2m} - \frac{2q-1}{2n} - N\right),$$

the corresponding structure function being given by:

$$\begin{aligned}
\Phi(E, x) = \Phi^N_{(p,q)}(x) = & \\
= \prod_{k=1}^{m}\left(x + \frac{2p-1}{2m} - \frac{2k-1}{2m}\right) &\prod_{\ell=1}^{n}\left(N - x + \frac{2q-1}{2n} + \frac{2\ell-1}{2n}\right) \\
= \frac{1}{m^m n^n}\frac{\Gamma(mx+p)}{\Gamma(mx+p-m)}&\frac{\Gamma((N-x)n+q+n)}{\Gamma((N-x)n+q)}.
\end{aligned}$$
(13)

In all these equations one has $N = 0, 1, 2, \ldots$, while the dimensionality of the representation is given by $N+1$. Eq. (12) means that there are $m \cdot n$ energy eigenvalues corresponding to each N value, each eigenvalue having degeneracy $N + 1$. (Later we shall see that the degenerate states corresponding to the same eigenvalue can be labelled by an "angular momentum".) The energy formula can be corroborated by using the corresponding Schrödinger

equation. For the Hamiltonian of eq. (1) the eigenvalues of the Schrödinger equation are given by:

$$E = \frac{1}{m}\left(n_x + \frac{1}{2}\right) + \frac{1}{n}\left(n_y + \frac{1}{2}\right), \tag{14}$$

where $n_x = 0, 1, \ldots$ and $n_y = 0, 1, \ldots$. Comparing eqs (12) and (14) one concludes that:

$$N = [n_x/m] + [n_y/n],$$

where $[x]$ is the integer part of the number x, and

$$p = \mathrm{mod}(n_x, m) + 1, \quad q = \mathrm{mod}(n_y, n) + 1.$$

The eigenvectors of the Hamiltonian can be parametrized by the dimensionality of the representation N, the numbers p, q, and the number $k = 0, 1, \ldots, N$:

$$H \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle = \left(N + \frac{2p-1}{2m} + \frac{2q-1}{2n}\right) \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle, \tag{15}$$

$$S_0 \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle = \left(k + \frac{1}{2}\left(\frac{2p-1}{2m} - \frac{2q-1}{2n} - N\right)\right) \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle, \tag{16}$$

$$S_+ \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle = \sqrt{\Phi^N_{(p,q)}(k+1)} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k+1 \right\rangle, \tag{17}$$

$$S_- \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle = \sqrt{\Phi^N_{(p,q)}(k)} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k-1 \right\rangle. \tag{18}$$

It is worth noticing that the operators $S_0, S_\pm$ do not correspond to a generalization of the angular momentum, S_0 being the operator corresponding to the Fradkin operator $S_{xx} - S_{yy}$ [6,7]. The corresponding "angular momentum" is defined by:

$$L = -i(S_+ - S_-). \tag{19}$$

The "angular momentum" operator commutes with the Hamiltonian:

$$[H, L] = 0.$$

Let $|\ell>$ be the eigenvector of the operator L corresponding to the eigenvalue ℓ. The general form of this eigenvector can be given by:

$$|\ell> = \sum_{k=0}^{N} \frac{i^k c_k}{\sqrt{[k]!}} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle. \qquad (20)$$

In order to find the eigenvalues of L and the coefficients c_k we use the Lanczos algorithm [22], as formulated in [23]. From eqs (17) and (18) we find

$$L|\ell> = \ell|\ell> = \ell \sum_{k=0}^{N} \frac{i^k c_k}{\sqrt{[k]!}} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle =$$

$$= \frac{1}{i} \sum_{k=0}^{N-1} \frac{i^k c_k \sqrt{\Phi_{(p,q)}^{N}(k+1)}}{\sqrt{[k]!}} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k+1 \right\rangle - \frac{1}{i} \sum_{k=1}^{N} \frac{i^k c_k \sqrt{\Phi_{(p,q)}^{N}(k)}}{\sqrt{[k]!}} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k-1 \right\rangle \qquad (21)$$

From this equation we find that:

$$c_k = (-1)^k 2^{-k/2} H_k(\ell/\sqrt{2}),$$

where the function $H_k(x)$ is a generalization of the "Hermite" polynomials, satisfying the recurrence relations:

$$H_{-1}(x) = 0, \quad H_0(x) = 1,$$

$$H_{k+1}(x) = 2x H_k(x) - 2\Phi_{(p,q)}^{N}(k) H_{k-1}(x),$$

and the "angular momentum" eigenvalues ℓ are the roots of the polynomial equation:

$$H_{N+1}(\ell/\sqrt{2}) = 0.$$

Therefore for a given value of N there are $N+1$ "angular momentum" eigenvalues ℓ, symmetric around zero (i.e. if ℓ is an "angular momentum" eigenvalue, then $-\ell$ is also an "angular momentum" eigenvalue). In the case of the symmetric harmonic oscillator ($m/n = 1/1$) these eigenvalues are uniformly distributed and differ by 2. In the general case the "angular momentum" eigenvalues are non-uniformly distributed. Remember that to each value of N correspond $m \cdot n$ energy levels, each with degeneracy $N+1$.

In conclusion, the two-dimensional anisotropic harmonic oscillator with rational ratio of frequencies equal to m/n is described dynamically by a non-linear extension of the u(2) Lie algebra, the order of this algebra being $m+n-$

1. The representation modules of this algebra can be generated by using the deformed oscillator algebra. The energy eigenvalues are calculated by the requirement of the existence of finite dimensional representation modules. The deformed algebra found here to be the symmetry algebra of the planar anisotropic quantum harmonic oscillator with rational ratio of frequencies answers the question posed originally by Jauch and Hill [3] in 1940. The next step should be the extension of the proposed algebra to the three-dimensional and multi-dimensional anisotropic oscillator.

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