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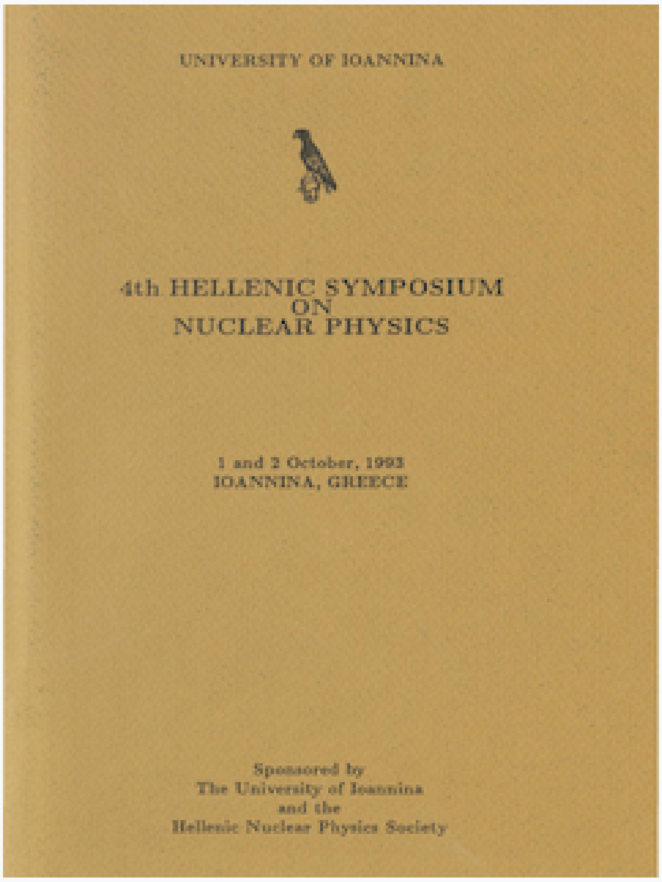

Generalized deformed SU(2) algebras in Nuclear Physics

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GENERALIZED DEFORMED SU(2) ALGEBRAS IN NUCLEAR PHYSICS [1]

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Abstract

A generalized deformed algebra $SU_\Phi(2)$, characterized by a structure function Φ, is obtained. The usual SU(2) and $SU_q(2)$ algebras correspond to specific choices of the structure function Φ. The action of the generators of the algebra on the relevant basis vectors, as well as the eigenvalues of the Casimir operator, are easily obtained. Possible applications in improving phenomenological nuclear models are discussed.

1. Introduction

Quantum algebras [1,2,3,4] (also called quantum groups) are nonlinear deformations of the corresponding Lie algebras, to which they reduce when the deformation parameter q is set equal to one. They are recently finding several applications in physics, especially after the introduction of the q-deformed harmonic oscillator [5,6]. Initially used for solving the quantum Yang-Baxter equation [7], they are now used in conformal field theory [8,9], quantum gravity [10,11], quantum optics [12,13,14,15], supersymmetric quantum mechanics [16], superintegrable systems [17], as well as in the description of spin chains [18,19], while in atomic physics attention has been focused on the hydrogen atom [20,21].

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One of the earliest applications of quantum algebras in physics was the use of the $SU_q(2)$ symmetry for the description of rotational spectra of deformed nuclei [22,23,24], superdeformed nuclei [25] and diatomic molecules [26,27,28], as well as for the description of the electromagnetic transition probabilities connecting these levels [29]. It has been early realized [23] that the deformation parameter τ^2 (with $q = e^{i\tau}$) is connected to the softness parameter of the Variable Moment of Inertia (VMI) model, so that the deformation of the SU(2) algebra is a way of taking into account the streching effect in rotating nuclei and molecules.

The introduction of the $SU_q(2)$ model has stimulated much work in the direction of applications of quantum algebras in nuclear and molecular physics. In molecular physics attention has been focused on the description of vibrational spectra of diatomic and polyatomic molecules in terms of deformed oscillators [30,31,32,33,34,35,36,37], as well as on the construction of equivalent potentials giving the same spectrum as these oscillators [38,39,40,41,42]. In nuclear physics correlated fermion pairs in a single-j nuclear shell model have been described in terms of deformed bosons [43,44,45], and deformed versions of various exactly soluble nuclear models, such as the toy Interacting Boson Model in two dimensions, the Moszkowski model, the Lipkin model, have been constructed [46,47,48,49], in an effort to understand the effects of deformation on well known models.

Although the $SU_q(2)$ model gives results which are in very good agreement with experiment [22,23,25,26,27,28,29], improvements seem to be possible [50]. One way to try to improve the model is by introducing a generalized deformed SU(2) symmetry, possibly containing the $SU_q(2)$ symmetry as a special case. Such a symmetry is constructed in the present paper. It should be noticed that although several versions of generalized deformed oscillators [51,52,53], as well as unification schemes for them, have been introduced (see [16] for references), the study of generalized deformed algebras has just started [54,55,56,57].

In the present work we construct a generalized deformed SU(2) algebra, characterized by a structure function Φ. The usual SU(2) and $SU_q(2)$ algebras are obtained for specific forms of the structure function, but additional forms are possible. The present method allows for the determination of the action of the generators on the basis vectors and of the eigenvalues of the Casimir operator in a simple way. Its possible usefulness in physical applications is also discussed.

2. Construction of a generalized deformed SU(2) algebra

We start the construction of the algebra in a very general way, adding the necessary restrictions as we proceed. Consider a Hilbert space V, consisting of the tensor sum of the subspaces V_l, i.e.

$$V = \oplus_{l=0}^{\infty} V_l, \tag{1}$$

where the subspaces V_l are unitary subspaces of dimension $2l+1$ and basis vectors $|l, m>$ with l integer or half-integer (in what follows we will denote the set of integers and half-integers by I_l) and m taking values from the set $S_l = \{-l, -l+1, \ldots, l-1, l\}$. The basis vectors are orthonormal,

$$< l', m'|l, m >= \delta_{ll'}\delta_{mm'},$$

and cover V

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} |l, m >< l, m| = 1.$$

In V we consider the operators J_0, J_+, J_-, the action of which on the basis vectors is given by

$$J_0|l, m >= m|l, m >, \quad m \in S_l, \quad l \in I_l, \tag{2}$$

$$J_+|l, m >= A(l, m)|l, m+1 >, \quad m \in S_l, \quad l \in I_l, \tag{3}$$

$$J_+|l, l >= 0, \tag{4}$$

$$J_- = (J_+)^{\dagger}, \tag{5}$$

where $A(l, m)$ is a real entire function defined for $m \in [-l, l]$, $l \in [0, \infty)$, satisfying the equations

$$A(l, l) = 0, \tag{6}$$

$$A(l, -l-1) = 0, \tag{7}$$

It is clear that of interest are the values of $A(l, m)$ with $l \in I_l$, $m \in S_l - \{l\}$.

¿From eqs (2)–(5) we immediately obtain

$$J_0 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} m|l, m >< l, m|, \tag{8}$$

$$J_+ = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A(l,m)|l,m+1><l,m|, \tag{9}$$

$$J_- = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A(l,m)|l,m><l,m+1|. \tag{10}$$

Using eqs (8)-(10) one can easily prove that

$$[J_0, J_+] = J_+, \tag{11}$$

$$[J_0, J_-] = -J_-, \tag{12}$$

$$J_0^n J_+ = J_+(J_0+1)^n, \tag{13}$$

$$J_0^n J_- = J_-(J_0-1)^n. \tag{14}$$

Then for every entire function A one has

$$A(J_0)J_+ = J_+A(J_0+1), \tag{15}$$

$$A(J_0)J_- = J_-A(J_0-1). \tag{16}$$

¿From eqs (9)–(10) one easily obtains

$$J_+J_- = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (A(l,m-1))^2|l,m><l,m|, \tag{17}$$

$$J_-J_+ = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (A(l,m))^2|l,m><l,m|. \tag{18}$$

One can define an operator J such that

$$J|l,m>= l|l,m>, \quad m \in S_l, \quad l \in I_l. \tag{19}$$

Clearly one has

$$J = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} l|l,m><l,m|, \tag{20}$$

and

$$(A(J,J_0))^2|l,m>= (A(l,m))^2|l,m> . \tag{21}$$

Then eqs (17)-(18) can be written as

$$J_+J_- = (A(J,J_0-1))^2, \tag{22}$$

$$J_-J_+ = (A(J, J_0))^2. \tag{23}$$

For the commutator of J_+ with J_- one has from eqs (17), (18) that

$$[J_+, J_-] = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left((A(l, m-1))^2 - (A(l, m))^2\right) |l, m >< l, m|, \tag{24}$$

while from eqs (22), (23) one finds

$$[J_+, J_-] = (A(J, J_0 - 1))^2 - (A(J, J_0))^2. \tag{25}$$

In what follows we wish to restrict ourselves to operators J_0, J_+, J_- which close an algebra by themselves, i.e. without involving J. Eqs (11), (12) already do not involve J, but eq. (25) does. We wish to restrict ourselves to algebras for which the right hand side (rhs) of eq. (25) is a function of J_0 only. We assume that this function of J_0 can be written in the form $B(J_0) - B(J_0 - 1)$, i.e. we require that

$$[J_+, J_-] = B(J_0) - B(J_0 - 1). \tag{26}$$

(For sufficient conditions under which a function of J_0 can be written in the form $B(J_0) - B(J_0 - 1)$ see ref. [58].) By equating the rhs of eqs (25) and (26) and acting on the basis vector $|l, m >$ we find that for every $m \in S_l$ and $l \in I_l$ the following condition should be satisfied

$$(A(l, m))^2 - (A(l, m-1))^2 = B(m-1) - B(m). \tag{27}$$

This condition is satisfied if $(A(l, m))^2$ is separable into the difference of a function of l and a function of m, i.e.

$$(A(l, m))^2 = C(l) - B(m), \quad m \in S_l, \quad l \in I_l. \tag{28}$$

¿From eqs (6), (7) and (28) one easily sees that

$$C(l) = B(l), \tag{29}$$

and

$$B(l) = B(-l - 1). \tag{30}$$

The last equation implies that $B(l)$ is of the form

$$B(l) = \Phi(l(l+1)). \tag{31}$$

This can be proven as follows: In eq. (30) one can put $l = j - \frac{1}{2}$. Then $B(j - \frac{1}{2}) = B(-j - \frac{1}{2})$. Thus the function $G(j) = B(j - \frac{1}{2})$ is an even function of j, i.e. $G(j) = G(-j)$. For every even function $G(j)$ one can find a function F such that $G(j) = F(j^2)$, which implies that $B(l) = F(l^2 + l + \frac{1}{4})$. As a result there is a function $\Phi(x) = F(x + \frac{1}{4})$, for which eq. (31) is valid. (The inverse also holds: For every function of the form given in eq. (31), eq. (30) is satisfied, as one can trivially see.)

¿From eqs. (28), (29) and (31) one then finds that

$$(A(l,m))^2 = \Phi(l(l+1)) - \Phi(m(m+1)), \quad m \in S_l, \quad l \in I_l. \tag{32}$$

This can be written as

$$(A(J,J_0))^2 = \Phi(J(J+1)) - \Phi(J_0(J_0+1)), \tag{33}$$

because of the following general proposition:

$$F(l,m) = 0, \quad m \in S_l, l \in I_l \Leftrightarrow F(J,J_0) = 0, \tag{34}$$

where $F(l,m)$ is any entire function. The proof of the proposition is simple. From $F(l,m) = 0$ one has $F(l,m)|l,m >= 0$ and then $F(J,J_0)|l,m >= 0$ for every $m \in S_l$ and $l \in I_l$, which implies that $F(J,J_0) = 0$. The inverse is also proved through the same steps.

¿From eq. (33) it is clear that $\Phi(x)$ must be an increasing function for $x > 0$. Thus the restricted as described above algebra satisfies the relations

$$[J_0, J_+] = J_+, \qquad [J_0, J_-] = -J_-, \tag{35}$$

$$J_-J_+ = \Phi(J(J+1)) - \Phi(J_0(J_0+1)), \tag{36}$$

$$J_+J_- = \Phi(J(J+1)) - \Phi(J_0(J_0-1)), \tag{37}$$

$$[J_+, J_-] = \Phi(J_0(J_0+1)) - \Phi(J_0(J_0-1)), \tag{38}$$

where $\Phi(x)$ is any increasing entire function defined for $x \geq -\frac{1}{4}$. This algebra is a generalization of SU(2), characterized by the function Φ. Therefore we are going to use for this the symbol $SU_\Phi(2)$.

Using eqs (35)-(38) one can easily verify that the Casimir operator (which commutes with all the generators of the algebra) is

$$C = J_- J_+ + \Phi(J_0(J_0+1)) = J_+ J_- + \Phi(J_0(J_0-1)). \tag{39}$$

¿From eq. (36) one then has

$$C = \Phi(J(J+1)). \tag{40}$$

¿From this equation it is clear that the eigenvalues of the Casimir operator in the basis $|l, m>$ are $\Phi(l(l+1))$, with $l = 0,\ 1/2,\ 1,\ 3/2,\ \ldots$. The action of the various operators on the basis vectors is summarized by

$$J_0|l,m>= m|l,m>, \tag{41}$$

$$J_+|l,m>= \sqrt{\Phi(l(l+1)) - \Phi(m(m+1))}\,|l,m+1>, \tag{42}$$

$$J_-|l,m>= \sqrt{\Phi(l(l+1)) - \Phi(m(m-1))}\,|l,m-1>, \tag{43}$$

$$C|l,m>= \Phi(l(l+1))|l,m> . \tag{44}$$

We have therefore completed the construction of an algebra $SU_\Phi(2)$, which is a generalization of the SU(2) algebra characterized by the *structure function* Φ.

3. Discussion

Several comments on the mathematical and physical implications of the $SU_\Phi(2)$ algebra of the previous section are now in place:

i) The usual SU(2) algebra is obtained for

$$\Phi(x(x+1)) = x(x+1),$$

as one can see from eqs. (35)-(44).

ii) The quantum algebra $SU_q(2)$, with commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = [2J_0]_q, \tag{45}$$

is obtained for

$$\Phi(x(x+1)) = [x]_q [x+1]_q, \tag{46}$$

with q-numbers defined as

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

One can be persuaded that the function $\Phi(x(x+1))$ given in eq. (46) is really a function of the variable $x(x+1)$ (a fact that is not immediately obvious) by having a look at the Taylor expansions given in eqs. (10a) and (10b) of ref. [23].

iii) In ref. [54] the following formalism is used:

$$[J_+, J_-] = f(J_0), \qquad C = J_- J_+ + h(J_0), \tag{47}$$

and the condition

$$f(J_0) = h(J_0) - h(J_0 - 1) \tag{48}$$

is found to hold. Similar formalisms have been used in [55,56,57]. These results correspond to

$$h(J_0) = \Phi(J_0(J_0+1)), \qquad f(J_0) = \Phi(J_0(J_0+1)) - \Phi(J_0(J_0-1)),$$

which automatically satisfy the condition of eq. (48). In the present method the extra results of eqs (42)-(44) are obtained at no toil.

iv) It is clear that the rhs of eq. (38) is an odd function of J_0. This imposes an extra restriction on $f(J_0)$ of the previous formalism (eq. (47)), while it is automatically satisfied in the case of $SU_q(2)$, as one can easily see in eq. (45).

v) In ref. [50] it has been argued that the Hamiltonian

$$E(J) = a\left[\sqrt{1 + bJ(J+1)} - 1\right] \tag{49}$$

gives better agreement to rotational nuclear spectra than the one coming from the $SU_q(2)$ symmetry [22,23]. Using the present technique one can construct an $SU_\Phi(2)$ algebra giving the spectrum of eq. (49) exactly. This algebra is characterized by the structure function

$$\Phi(J(J+1)) = a\left[\sqrt{1 + bJ(J+1)} - 1\right].$$

It is of interest to check if this choice of structure function also improves the agreement between theory and experiment in the case of the electromagnetic transition probabilities connecting these energy levels. In order to study this problem, one has to construct the relevant generalized Clebsch-Gordan coefficients [29]. Work in this direction is in progress.

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