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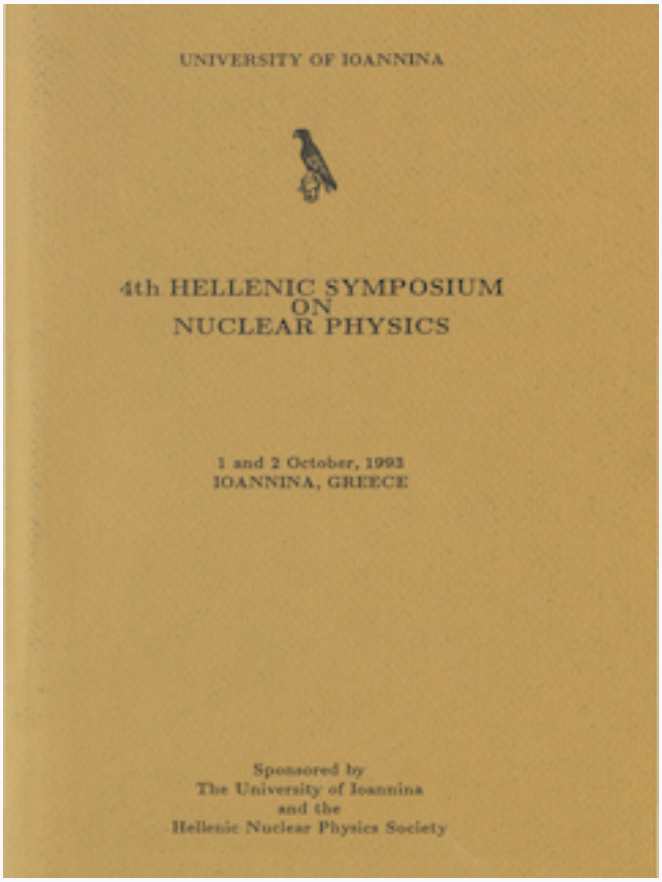

Strong interaction effects in Σ- atoms

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STRONG INTERACTION EFFECTS IN Σ^--ATOMS

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Abstract: An approximation is made for the Σ^- atomic energy shifts and widths due to strong interaction central and spin-orbit parts of the potential with the use of *the S-Matrix Approach*, introduced by Deloff. This simplified model is applied in $\Sigma - {}^{12}C$ in connection with the Σ-hypernuclear data: ${}^{12}_{\Sigma}C$. It is concluded that there is an essential influence of the strong interaction and especially of the nuclear spin-orbit coupling in the spectra of $\Sigma - {}^{12}C$, mainly for the p-shell.

1. Introduction

The Σ^--atoms belong to the family of exotic atoms. An exotic atom is an atom where a negative particle other than an electron is captured into an atomic orbit around a nucleus. Since electron is the least massive of the known negatively charged particles, the resulting exotic atomic states have smaller radii and larger binding energies than the electronic atomic states of the same quantum numbers. The heavy particle may be a lepton, a meson or a baryon [1].

The study of exotic atoms provides useful information for hadrons and for atomic nuclei which participate in the reaction and also for the interactions between them. The heaviest particle observed so far in an atomic orbit is Σ^--hyperon. The study of Σ^- hyperonic atoms is very interesting and can be used for the measurement of Σ^- magnetic moment, the study of nuclear properties and the understanding of ΣN interaction and generally of the hyperon-nucleon interaction [2].

The lifetime of Σ^--hyperons is 1.49 x 10^{-10} sec, nearly 100 times shorter than that of K^- mesons, so one cannot have a stopped Σ^- beam because Σ^- particles should decay before reaching the target-nucleus. So, for the creation of Σ^- atoms, beams of K^- mesons are used. When such a beam is stopped in a target, the K^- absorption is followed in some cases by the emission of low-energy Σ^--hyperons. The predominant elementary reactions which give rise to a Σ^--hyperon are [3]:

$$K^- + p \rightarrow \Sigma^- + \pi^+$$

$$K^- + n \rightarrow \Sigma^- + \pi^o$$

$$K^- + p + n \rightarrow \Sigma^- + p$$

The Σ^--hyperons produced by the first two reactions have energies about 20 MeV. The Σ^--hyperons after being produced are emitted from the nucleus. Since the lifetime of the Σ^--hyperons is comparable with their slowing down time, a part of them decays before forming a Σ^--hyperonic atom. The third reaction yields less than 30% of the totally produced Σ^--hyperons. Since these Σ^--hyperons have very high energies, most of them decay before they are captured, so the third reaction does not contribute to the formation of Σ^- atoms [3].

The Σ^--hyperons that are produced by the first two reactions and do not decay, are captured into atomic orbits forming Σ^--atoms. This process is not a simple one, but follows some discrete stages. Initially the Σ^--hyperons have large kinetic energies and penetrate the electronic cloud without managing being captured by the atomic orbits. They lose their energy gradually and they take velocities comparable with the velocities of the atomic electrons. So, an atomic capture can occur and have a bound state [3].

The capture of Σ^--hyperons takes place in atomic levels with high main quantum number n. After the capture of the Σ^--hyperons and the formation of the Σ^--atoms, a deexcitation follows and cascade of the Σ^--hyperons in the inner orbits. This cascade is accompannied by the emission of an Auger-electron to which the energy difference between the two levels is transferred. This energy is usually enough to take the electron from the electronic cloud [3, 4].

The emission of Auger-electrons is accompanied by X-ray emission, but the Auger phenomenon dominates. This situation is inversed when the orbital radius of the Σ^--hyperon becomes less than that of the innermost orbital electron [1]. Then we have only X-ray transitions. The electrons lie outside the hyperonic orbits, so they have very little effect on them; the same happens with the nucleus in this phase. So it becomes obvious that in this phase *the electromagnetic forces* dominate [1, 3, 4].

The X-ray emission takes on with the same rate, until an overlap happens with the nuclear matter distribution. In this last phase, the influence of the strong interaction changes the X-ray transitions. In the atomic state that the effects of *the strong interaction* start to happen and also in the following states, we have a variation in the intensity of the X-ray transition of the spectrum. This change occurs in the shift and the width of the spectral line [1, 2, 3, 4].

Experimental Data

The first clear evidence for the observation of Σ^--atoms was obtained at CERN in 1970 in three elements: S, Cl and Zn [3]. In this experiment X-ray transitions in these elements were measured, as well as the number of Σ^--atoms produced per stopped K^-. Σ/K^-. For ^{16}S all the transitions from $n = 6$ to $n = 4$ were measured, for ^{17}Cl from $n = 7$ to $n = 4$ and for ^{30}Zn from $n = 11$ to $n = 5$. There was an agreement between the measured and the calculated values of energy [3].

The second experiment took place at CERN in 1975 [2]. They observed X-ray transitions in Σ^--atoms in eight elements: C, P, Ca, Ti, Zn, Nb, Cd, Ba (Z varies from 6 to 56). They measured the X-ray intensities in these cases (for Σ^- atoms and K^- atoms), as well as the production of Σ^- hyperons per K^-. For four elements: C, Ca, Ti, Ba the yields of X-ray transitions due to strong interaction were measured analytically. There were also some theoretical calculations in order to reproduce the data.

The *yield* is defined by the ratio:

$$Y = \frac{\Gamma_x}{\Gamma_x + \Gamma_s},$$

In this formula Γ_x is the electromagnetic interaction in the $[(n, \ell)$ level] and Γ_s is the strong interaction in the $[(n+1, \ell+1)$ level].

The third experiment took place at the Rutherford Laboratory in 1978 [5]. It is the first direct measurement of X-ray shifts, widths and yields of Σ^--atoms in five elements: *O, Mg, Al, Si, S*. In that experiment there was the first measurement of the width Γ_s due to the strong interaction. There was a fitting of the experimental values with the use of an optical potential model and the value of the effective complex scattering length $\bar{a}$ was calculated.

Also in an experiment done in 1987 at BNL for the measurement of the magnetic dipole moment of the Σ^- hyperon, they measured X-ray spectra of Σ^--atoms in two elements: Pb and W. In these experiments strong interaction effects were also observed [6]. The effects of strong interaction in Σ^--atoms of Pb, W for the transition $(n = 10 \rightarrow n = 9)$ were also studied analytically at BNL, where the shifts and the widths for this case were measured [1].

We must point out that the possibility of formation of Σ^--atoms (Σ/K^-) for the heavy elements is about half the possibility for the light elements $(3\% - 8\%$ respectively) [7].

Reviewing the experimental data for Σ^--atoms, there are 5 measurements of shifts, 3 direct measurements of widths and 9 measurements of relative yields [8].

The Batty potential

The description of strong interaction effects in exotic atoms was done by Batty by using an optical complex potential of the form [5, 8, 9]:

$$U(r) = -\left(\frac{2\pi\hbar^2}{\mu}\right)\left(1 + \frac{m}{M}\right)\bar{a}\,\rho(r)$$

In the above formula μ is the reduced mass of the system Σ-nucleus, m is the mass of the Σ^- particle, M is the mass of the nucleon, $\bar{a}$ is the effective complex scattering length and $\rho(r)$ is the nucleon density distribution.

The calculations have given for $\bar{a}$:

$$\bar{a} = (0.35 \pm +0.004) + i(0.19 + 0.03) fm$$

For the density distribution $\rho(r)$ a Fermi form was used:

$$\rho(r) = \frac{\rho_o}{1 + exp(\frac{r-c}{z})}$$

$$\rho_o = 0.17\,, \qquad c = 6.624\,, \qquad z = 0.549$$

There is a difference between the above calculated value of $\bar{a}$ and the value of the scattering length which corresponds to the free ΣN interaction [10], (the $Im\bar{a}$ is smaller for the Σ atoms). An explanation for that is that a possible $\Sigma^- p$ bound state causes an effect of the Pauli exclusion principle in the scattering length [11].

Finally the following potential was found:

$$U(r) = \frac{(-27.8 - 15.5)}{1 + exp(\frac{r-c}{z})}\, MeV$$

This potential which was calculated from Σ^--atoms was used for the determination of the width of Σ^--hypernuclei. So, it was found that: $\Gamma \simeq 30\, MeV$ [12, 13, 14]. This is very important for Σ^--hypernuclei, because initially all authors used for their study the Batty potential expecting to find the value of about 30 MeV for the width. But this is not consistent with the experimental data, where the width was found to be about 4-7 MeV [14].

The existence of narrow Σ hypernuclear resonances was the puzzle of Σ^--hypernuclei. The Σ hypernuclear states should have large width, because the conversion $\Sigma N \rightarrow \Lambda N$ takes part. But the most important reason for that is the fitting of Σ atomic data done by Batty. If Batty took into consideration some other factors, he could have different results.

2. Calculation of the atomic level energy shifts and widths

In this chapter we will examine the effects of the nuclear spin-orbit coupling in Σ-atoms. This idea was also used by Bogdanova et al. [15] and it was concluded that there are sufficient effects in Σ-atoms due to the nuclear spin-orbit coupling.

In order to calculate the effects of the nuclear spin-orbit coupling within our model, we consider the potential:

$$V_{int}(r) = -\frac{Ze^2}{r} + (\,V + iW\,) - V_{so}(\bar{l}\cdot\bar{s})r_o^2\,\frac{1}{R}\,\delta(r-R)$$

The first term corresponds to the Coulomb potential, the second to the nuclear central potential and the third term corresponds to the spin-orbit potential.

Analytically, the spin-orbit potential is given by the formula:

$$V_{so}(r) = -V_{so}(\bar{l}\cdot\bar{s})\,r_o^2\,\frac{1}{r}\,\frac{dw(r)}{dr}$$

$$w(r) = \begin{cases} 1, & r < R \\ & R = r_o A^{1/3} \\ 0, & r \geq R \end{cases}$$

The spin-orbit forces act to a nucleon only to the surface r = R of the nucleus, because in this point there is a discontinuity in the potential. So, $V_{so}(r)$ takes the form:

$$V_{so}(r) = -V_{so}(\bar{l} \cdot \bar{s})\, r_o^2 \frac{1}{R}\, \delta(r - R)$$

The central line of the spectrum which corresponds to the Coulomb potential is shifted by the other two interactions.

(i) First order pertubation theory

The first order pertubation theory was used for the calculation of ε and Γ in hadronic atoms, but without success, because the strong interaction between the hyperon and the nucleon is very large [16].

The widths that correspond to the nuclear potential and to the spin-orbit potential, are taken by the formula:

$$\Gamma = \int_0^\infty \mid \Psi_c \mid^2 \; Im\, V(r)\, dr \tag{1}$$

The function Ψ_c is the Coulomb eigenfunction which corresponds to the solution of the Schrödinger equation for the Coulomb potential:

$$\frac{d^2\, \Psi_c(r)}{dr^2} + \left\{ \frac{2m}{\hbar^2} E + \frac{2m}{\hbar^2} \frac{Ze^2}{r} - \frac{\ell(\ell+1)}{r^2} \right\} \Psi_c(r) = 0$$

The wave function Ψ_c for circular orbits $n = \ell + 1$ has the form:

$$\Psi_\ell^c(r) = -\left\{ \left(\frac{2Z}{(\ell+1)a_o} \right)^3 \frac{1}{2(\ell+1)\,[\,(2\ell+1)\,!\,]^3} \right\}^{\frac{1}{2}} e^{-\frac{1}{2}\rho}\, \rho^{\ell+1}\, (-1)^{2\ell+1}\, (2\ell+1)\,!$$

$$a_o = \frac{\hbar^2}{\mu\, e^2}\,, \qquad \rho = \frac{2Z}{n\, a_o}\, r = -2\, i\, k_o\, r\,, \qquad k_o = i\, \frac{\mu\, Z\, e^2}{n\, \hbar^2}$$

k_o are the eigenvalues of the Coulomb problem.

$$\text{It is:} \qquad \Psi_c(r) = A\, e^{-\frac{\rho}{2}}\, \rho^{\ell+1}(r)$$

where A expresses the product of all the terms independent of r.

Next we will calculate the widths Γ_{centr} and Γ_{ls} that correspond to the central potential and the spin-orbit potential by using the equation (1).

$$| \Psi_\ell^c(r) |^2 = A^2 e^{-\rho} \rho^{2\ell+2}(r)$$

$$\Gamma_{centr} = \int_0^R A^2 e^{-\rho} \rho^{2\ell+2}(r) \, Im \, V_o \, dr$$

$$= -\frac{A^2}{2 i k_o} Im \, V_o \int_0^{-2ik_oR} e^{-\rho} \rho^{2\ell+2} \, d\rho$$

The above integral is given by the formula:

$$I = \int_0^{-2ik_oR} e^{-\rho} \rho^{2\ell+2} \, d\rho = \gamma (2\ell + 3 , -2ik_oR)$$

The function $\gamma(a, x)$ is the incomplete gamma function, which is defined by the formula:

$$\int_0^x e^{-t} t^{a-1} \, dt = \gamma (a, x)$$

So we have:

$$\Gamma_{centr} = - \frac{A^2}{2ik_o} Im \, V_o \gamma (2\ell + 3 , -2ik_oR)$$

For the quantity Γ_{ls} we have:

$$\Gamma_{ls} = \int_0^\infty | \Psi_c |^2 \; Im \, V_{ls} \, dr$$

After the suitable substitutions we take:

$$\Gamma_{ls} = A^2 e^{2ik_oR} (-2ik_oR)^{2\ell+2} \, \mathrm{Im}(V_{so}) \, (\bar{\ell} \cdot \bar{s}) \, \frac{r_o^2}{R}$$

The first order pertubation theory does not give good results (the result is two or three times the real value or has the opposite sign). This fact makes the introduction of another method necessary [16].

(ii) The Deloff Approximation

As was told, the strong interaction causes a variation in the energy eigenvalue. This variation δE is complex because it refers to both the shift ε and the width Γ. This complex shift δE is due to the nuclear central potential which consists of two terms: one that corresponds to the nuclear central potential and the other that corresponds to the spin-orbit coupling.

The quantities ε and Γ are much smaller than the Coulomb binding energy, so we can use an approximate method in order to calculate them. Next we will develop such a method which was used by Deloff and is called *the S-Matrix Approach* [16]. This method provides a fast and accurate evaluation of the complex

level shifts and gives values comparable with the exact values, as was shown by Deloff [16].

We consider that every bound state of a Σ atom is characterised by the quantum numbers (n, ℓ). If E and E_o are the energy eigenvalues with and without the strong interaction, the variation δE is given by the relation:

$$\delta E = E - E_o = -\epsilon - i\frac{\Gamma}{2}$$

We consider the equation:

$$\frac{d^2 R_\ell(r)}{dr^2} + \left\{ \frac{2m}{\hbar^2}[E - V(r)] + \frac{2m}{\hbar^2}\frac{Ze^2}{r} - \frac{\ell(\ell+1)}{r^2} \right\} R_\ell(r) = 0$$

$$\text{Where:} \quad V(r) = \begin{cases} V_o, & r < R \\ V_o - V_{so}(\bar{l}\cdot\bar{s})r_o^2 \frac{1}{R}\delta(r-R), & r = R \\ 0, & r > R \end{cases}$$

If we set: $k^2 = 2mE/\hbar^2$, we take the equation:

$$R_\ell''(r) + \left\{ k^2 + \frac{2m}{\hbar^2}\frac{Ze^2}{r} - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2}V(r) \right\} R_\ell(r) = 0 \qquad (2)$$

In equation (2), m is the reduced mass of the system Σ-nucleus:

$$m = \frac{m_\Sigma A m_N}{m_\Sigma + A m_N}$$

In the above formula m_Σ is the mass of the Σ^- particle, m_N is the mass of a nucleon and A is the nuclear mass number.

Let the functions $\phi(k,r)$, $f(k,r)$ be the regular and the irregular solutions of equation (2). Also the functions $\phi^c(k,r)$, $f^c(k,r)$ are the pure Coulomb wavefunctions which are solutions of equation (2), when $V(r) = 0$.

Next we define the Jost function, F(k):

$$F(k) = W[f(k,r), \phi(k,r)] + \frac{2m}{\hbar^2}V_{so}(\bar{l}\cdot\bar{s})r_o^2\frac{1}{R}f(k,r)\phi(k,r)$$

When $r \geq R$: $f(k,r) = f^c(k,r)$, so we have:

$$F(k) = W[f^c(k,r), \phi(k,r)] + \frac{2m}{\hbar^2}V_{so}(\bar{l}\cdot\bar{s})r_o^2\frac{1}{R}f^c(k,r)\phi(k,r)$$

The solution of equation F(k) = 0 will give us the binding energy E_o.

We introduce the function:

$$H(k) = \frac{F(k)}{f^c(k,R)\phi(k,R)}$$

We will solve the equation: $H(k) = 0$.

$$H(k) = \frac{\phi'(k,R)}{\phi(k,R)} - \frac{f^{c'}(k,R)}{f^c(k,R)} + \frac{2m}{\hbar^2} V_{so}(\bar{l}\cdot\bar{s}) r_o^2 \frac{1}{R} = 0 \tag{3}$$

This equation will be solved with the Newton-Raphson method [16]. The initial value of the root κ of the above relation is defined by setting: $k = \kappa + \delta k$ and expanding the function H(k) in a Taylor series around κ:

$$H(k) = H(\kappa) + \delta k\, \dot{H}(\kappa) + \ldots$$

We omit the terms of higher order and we take:

$$H(\kappa) + \delta k\, \dot{H}(\kappa) = 0 \quad \Longrightarrow \quad \delta k = -\frac{H(\kappa)}{\dot{H}(\kappa)} \tag{4}$$

(With the dot we mean differentiation with respect to k).

Next we will calculate the function $\dot{H}(\kappa)$. We differentiate the equation (3) with respect to k and we set $k = \kappa$.

$$\dot{H}(\kappa) = \frac{W\left[\phi(\kappa,R), \dot{\phi}(\kappa,R)\right]}{\left[\phi(\kappa,R)\right]^2} - \frac{W\left[f^c(\kappa,R), \dot{f}^c(\kappa,R)\right]}{\left[f^c(\kappa,R)\right]^2} \tag{5}$$

We can get rid of the terms $\dot{f}^c(\kappa,R), \dot{\phi}(\kappa,R)$, by differentiating the initial equation (2) with respect to k and by setting: $R_\ell = \phi(k,r)$.

$$\dot{\phi}''(k,r) + \left\{k^2 + \frac{2m}{\hbar^2}\frac{Ze^2}{r} - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2}V(r)\right\}\dot{\phi}(k,r) = -2k\phi(k,r) \tag{6}$$

$$\phi''(k,r) + \left\{k^2 + \frac{2m}{\hbar^2}\frac{Ze^2}{r} - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2}V(r)\right\}\phi(k,r) = 0 \tag{2}$$

We multiply equation (2) by $\dot{\phi}(k,r)$, equation (6) by $\phi(k,r)$ and then by subtracting we have:

$$\phi(k,r)\,\dot{\phi}''(k,r) - \phi''(k,r)\,\dot{\phi}(k,r) = -2k\left[\phi(k,r)\right]^2$$

We add and subtract in the first part of this equation the term $\dot{\phi}'(k,r)\,\dot{\phi}'(k,r)$:

$$\frac{d}{dr}\left[\phi(k,r)\,\dot{\phi}'(k,r) - \dot{\phi}(k,r)\,\dot{\phi}'(k,r)\right] = -2k\left[\phi(k,r)\right]^2$$

$$\Longrightarrow \quad \frac{d}{dr}W\left[\phi(k,r),\, \dot{\phi}(k,r)\right] = -2k\left[\phi(k,r)\right]^2$$

In a similar way we calculate for the quantity $f^c(k,r)$:

$$\frac{d}{dr}W\left[f^c(k,r),\,\dot{f}^c(k,r)\right] = -2k\,[\,f^c(k,r)\,]^2$$

Now we can take the two Wronskians $W[\phi,\dot{\phi}]$ and $W[f^c,\dot{f}^c]$, by integrating the above relations in the limits $(0,R)$ and (R,∞) respectively:

$$W\left[\phi(\kappa,R),\,\dot{\phi}(\kappa,R)\right] = -2k\int_0^R [\,\phi(\kappa,r)\,]^2 dr$$

$$W\left[f^c(\kappa,R),\,\dot{f}^c(\kappa,R)\right] = -2k\int_R^\infty [\,f^c(\kappa,r)\,]^2 dr$$

We substitute these relations in equation (5) and we take:

$$\dot{H}(\kappa) = -2\kappa\left\{\int_0^R\left[\frac{\phi(\kappa,r)}{\phi(\kappa,R)}\right]^2 dr + \int_R^\infty\left[\frac{f^c(\kappa,r)}{f^c(\kappa,R)}\right]^2 dr\right\}$$

Since the range R is small, we can in a first approximation neglect the first term. So we take:

$$\dot{H}(\kappa) = -2\kappa\int_R^\infty\left[\frac{f^c(\kappa,r)}{f^c(\kappa,R)}\right]^2 dr$$

Since from equation (4) we have: $\delta k = -H(\kappa)/\dot{H}(\kappa)$, by substituting we take:

$$\delta k = \frac{\left\{W\left[f^c(\kappa,r),\phi(\kappa,R)\right] + \frac{2m}{\hbar^2}V_{so}(\bar{l}\cdot\bar{s})r_o^2\frac{1}{R}f^c(\kappa,R)\,\phi(\kappa,R)\right\}\,f^c(\kappa,R)}{2\kappa\,\phi(\kappa,R)\int_0^\infty[f^c(\kappa,r)]^2 dr} \qquad (7)$$

This equation is the basis of the iterative Newton-Raphson method. The number of iterations depends on the accuracy of the initial value. Since as was told the quantity δE is smaller than the Coulomb binding energy, if we take the Coulomb energy as an initial value, we can reach the desired value with only two iterations.

If $F^c(k)$ is the Jost solution for the Coulomb eigenfunctions:

$$F^c(k) = W[f^c(k,r),\phi^c(k,r)]$$

and k_o is the root of equation $F^c(k) = 0$, for $k = k_o$ the function $f^c(k_o,r)$ is analogous to the function $\phi^c(k_o,r)$.

Let be: $f^c(k_o,r) = a\phi^c(k_o,r)$. Then the formula (7) is simplified:

$$\delta k = \frac{1}{2k_o}\left\{\frac{\phi'(k_o,R)}{\phi(k_o,R)} - \frac{\phi'^c(k_o,R)}{\phi^c(k_o,R)} + \frac{2m}{\hbar^2}V_{so}(\bar{l}\cdot\bar{s})\frac{r_o^2}{R}\right\}\Big/\int_0^\infty\left[\frac{\phi^c(k_o,r)}{\phi^c(k_o,R)}\right]^2 dr$$

There is also a similar relation for δE: $\delta E = \frac{\hbar^2}{2m}2k\delta k$.

$$\delta E = \frac{\hbar^2}{2m}\left\{\frac{\phi'(k_o,R)}{\phi(k_o,R)} - \frac{\phi'^c(k_o,R)}{\phi^c(k_o,R)} + \frac{2m}{\hbar^2}V_{so}(\bar{l}\cdot\bar{s})\frac{r_o^2}{R}\right\} \Big/ \int_0^\infty \left[\frac{\phi^c(k_o,r)}{\phi^c(k_o,R)}\right]^2 dr \quad (8)$$

In Σ^- atoms and generally in hadronic atoms, we are interested in the circular orbits for which: $n = \ell + 1$. In this case, the Coulomb eigenvalues k_o are given by the relation:

$$E_n = -\frac{mZ^2e^4}{2\hbar^2 n^2} \Longrightarrow k_o = \frac{imZe^2}{n\hbar^2}$$

We must calculate the integral $\int_0^\infty [\phi^c(k_o,r)/\phi^c(k_o,R)]^2 dr$ and the ratio $\phi'^c(k_o,R)/\phi^c(k_o,R)$ and substitute them in equation (8).
The general form of the Coulomb wavefunction is:

$$\phi^c_{n\ell} = -\left\{\left(\frac{2Z}{na_o}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right\}^{\frac{1}{2}} e^{-\frac{1}{2}\rho}\,\rho^{\ell+1}\,L^{2\ell+1}_{n+\ell}(\rho) \quad (9)$$

In the previous equation we have: $a_o = \hbar^2/\mu e^2$, $\rho = -2ik_o r$.
The functions $L^{2\ell+1}_{n+\ell}$ are the Laguerre functions.
For the case that $n = \ell + 1$, the functions $L^{2\ell+1}_{n+\ell}$ are simplified:

$$L^{2\ell+1}_{2\ell+1}(\rho) = (-1)^{2\ell+1}\,(2\ell+1)!$$

It is clear that the function $L^{2\ell+1}_{n+\ell}$ is independent of r. We substitute in equation (9):

$$\phi^c_\ell = -\left\{\left(\frac{2Z}{(\ell+1)a_o}\right)^3 \frac{1}{2\,(\ell+1)\,[(2\ell+1)!]^3}\right\}^{\frac{1}{2}} e^{-\frac{1}{2}\rho}\,\rho^{\ell+1}\,(-1)^{2\ell+1}\,(2\ell+1)!$$

This function was also used in the previous subsection. As was told, we are interested in the quantities $\int_0^\infty [\phi^c(k_o,r)/\phi^c(k_o,R)]^2 dr$ and $\phi'^c(k_o,R)/\phi^c(k_o,R)$, so in the above relation we are interested only in the terms dependent on r. We introduce the constant A, which is equal with the product of all the independent of r terms.

$$\phi^c_\ell(r) = -A\,e^{ik_o r}\,(-2ik_o)^{\ell+1}\,r^{\ell+1}$$

We differentiate with respect to r, and finally we take the relation:

$$\frac{[\phi^c_\ell(r)]'}{\phi^c_\ell(r)} = ik_o + \frac{\ell+1}{r} \quad (10)$$

So we now have to calculate the integral: $\int_0^\infty [\phi^c(k_o,r)/\phi^c(k_o,R)]^2\,dr$ and to substitute in equation (8).

$$\int_0^\infty \left[\frac{\phi^c(k_o,r)}{\phi^c(k_o,R)}\right]^2 dr = e^{-2ik_oR}\,R^{-2\ell-2}\int_0^\infty e^{2ik_or}\,r^{2\ell+2}\,dr$$

Finally we have:

$$\int_0^\infty \left[\frac{\phi^c(k_o,r)}{\phi^c(k_o,R)}\right]^2 dr = \frac{R\,\Gamma(2\ell+3)}{exp(2ik_oR)\,(-2ik_oR)^{2\ell+3}} \tag{11}$$

We substitute the equations (11) and (10) into equation (8) and we take:

$$\delta E = \frac{\hbar^2}{2m}\frac{exp(2ik_oR)\,(-2ik_oR)^{2\ell+3}}{R\Gamma(2\ell+3)}$$
$$\left[\frac{\phi'(k_o,R)}{\phi(k_o,R)} - \frac{(\ell+1)}{R} - ik_o + \frac{2m}{\hbar^2}V_{so}\,(\bar{l}\cdot\bar{s})\,\frac{r_o^2}{R}\right]$$

By multiplying and dividing with R we take the relation:

$$\delta E = \frac{\hbar^2}{2m}\frac{exp(2ik_oR)\,(-2ik_oR)^{2\ell+3}}{R^2\Gamma(2\ell+3)}$$
$$\left[R\,\frac{\phi'(k_o,R)}{\phi(k_o,R)} - (\ell+1) - ik_oR + \frac{2m}{\hbar^2}V_{so}\,(\bar{l}\cdot\bar{s})\,r_o^2\right]$$

We divide this relation into two terms: one that depends on the spin-orbit coupling δE_1 and another that includes the rest terms δE_2:

$$\delta E = \delta E_1 + \delta E_2$$

$$\delta E_1 = \frac{exp(2ik_oR)\,(-2ik_oR)^{2\ell+3}}{R^2\,\Gamma(2\ell+3)}V_{so}\,(\bar{l}\cdot\bar{s})\,r_o^2 \tag{12}$$

$$\delta E_2 = \frac{\hbar^2}{2m}\frac{exp(2ik_oR)\,(-2ik_oR)^{2\ell+3}}{R^2\Gamma(2\ell+3)}\left[R\,\frac{\phi'(k_o,R)}{\phi(k_o,R)} - (\ell+1) - ik_oR\right] \tag{13}$$

(where: $k_o = imZe^2/n\hbar^2$).

The above relations are very simple; the shift δE_2 is expressed only in terms of the logarithmic derivative of the regular wavefunction at the nuclear boundary [16]. The wavefunction is given by the confluent hypergeometric functions.

3. Results and Conclusions

Next we will show some numerical results for $\Sigma^{-12}C$ for $\ell = 1, \ell = 2, \ell = 3$, (Tables 1, 2 and 3). We consider the case of ^{12}C, because only for this element there are atomic and hypernuclear data and the purpose of this paper is the study

of Σ-atoms in connection with Σ-hypernuclei. Since for $\ell > 1$, the quantities δE_1, δE_2 given by equations (12) and (13) are very small, we show only some representative results.

The potential model is the same with the one used by the same authors for the study of Σ-hypernuclei [17]. The value of the constant r_o ($R = r_o A^{1/3}$) is taken to be 1.31 fm [17]. The structure of the Σ-atomic wavefunctions is very sensitive on the value of r_o [18].

For $\ell = 1$, we take results for two values of the potential:

$$(i)\ V = (-12 - 4i)\ MeV, \quad (ii)\ V = (-16 - 2i)\ MeV$$

We also give some results for $\ell = 2, \ell = 3, \ell = 4$, for the potential (i).

Table 1
Energy Shifts in $\Sigma - {}^{12}C$
due to the spin-orbit coupling δE_1
and other quantities in MeV,
when V=(-12-i4) MeV, E_o*= -0.2589 MeV*
and δE_2 *= (-0.0128-i 0.0061) MeV.*

V_{so}	State	δE_1	δE_1 / δE_2	$\delta E_1/(E_o + \delta E_2)$
10	$p_{3/2}$	0.0037	-0.289	-0.014
20	$p_{3/2}$	0.0074	-0.580	-0.027
40	$p_{3/2}$	0.0147	-1.148	-0.054
50	$p_{3/2}$	0.0185	-1.445	-0.068
10	$p_{1/2}$	-0.0074	0.578	0.027
20	$p_{1/2}$	-0.0147	1.148	0.054
40	$p_{1/2}$	-0.0296	2.311	0.109
50	$p_{1/2}$	-0.0370	2.890	0.136

From the results of the tables 1, 2 and 3 we reach the following conclusions, which are valid only in the context of this simplified model:

(i) The shifts in the energy eigenvalue E_o (Coulomb energy) caused by the nuclear central potential and the spin-orbit potential are small for $\ell > 1$ and they tend to zero when ℓ is augmented. So only for $\ell = 1$ we can take the effects of the central potential and the spin-orbit potential in Σ-atoms.

Table 2
Energy Shifts in $\Sigma -^{12} C$ due to the spin-orbit coupling δE_1 and other quantities in MeV, when V=(-12-i4) MeV.

V_{so}	State	E_o	δE_1	δE_2	$\delta E_1/\delta E_2$	$\delta E_1/(E_o+\delta E_2)$
10	$d_{5/2}$	-0.1148	0.940×10^{-5}	-0.99×10^{-5}-i0.38×10^{-5}	-0.946	-0.822×10^{-4}
10	$d_{3/2}$	-0.1148	-0.140×10^{-4}	-0.99×10^{-5}-i0.38×10^{-5}	1.414	0.120×10^{-3}
20	$f_{7/2}$	-0.0646	0.980×10^{-8}	-0.26×10^{-8}-i0.89×10^{-9}	-3.769	-0.152×10^{-6}
20	$f_{5/2}$	-0.0646	-0.131×10^{-8}	-0.26×10^{-8}-i0.89×10^{-9}	0.504	0.202×10^{-6}
10	$g_{9/2}$	-0.0413	0.860×10^{-12}	-0.26×10^{-12}-i0.94×10^{-13}	-3.308	-0.208×10^{-10}

Table 3
Energy Shifts in $\Sigma -^{12} C$ due to the spin-orbit coupling δE_1 and other quantities in MeV, when V=(-16-i2) MeV, E_o= -0.2589 MeV and δE_2 = (-0.0203 - i 0.0041) MeV.

V_{so}	state	δE_1	$\delta E_1/\delta E_2$	$\delta E_1/(E_o+\delta E_2)$
10	$p_{3/2}$	0.0037	-0.182	-0.013
20	$p_{3/2}$	0.0074	-0.360	-0.027
30	$p_{3/2}$	0.0111	-0.545	-0.040
40	$p_{3/2}$	0.0148	-0.729	-0.053
50	$p_{3/2}$	0.0185	-0.911	-0.066
10	$p_{1/2}$	-0.0074	0.365	0.027
20	$p_{1/2}$	-0.0148	0.729	0.053
30	$p_{1/2}$	-0.0222	1.094	0.079
40	$p_{1/2}$	-0.0296	1.458	0.106
50	$p_{1/2}$	-0.0367	1.808	0.131

(ii) For $\ell = 1$ the quantities δE_1, δE_2 are of the same range and when V_{so} is augmented, δE_1 becomes greater than δE_2. For great V_{so}, δE_1 is of the same range with E_o.

(iii) For $\ell = 1$ the quantities δE_1, δE_2 shift the energy eigenvalue lightly. This shift is relatively important. So there is a considerable effect of the nuclear central potential and the spin-orbit coupling in Σ-atoms.

There must be a comparison with the exact values with this type of the potential like the one done by Deloff [16] in order to confirm these conclusions.

In this paper we have seen the importance of the third parameter of the central potential, the spin-orbit potential, in the spectra of Σ-atoms. The introduction of this third parameter of the potential in the fitting that gave the Batty potential is possible to change the values of the other two potential parameters, the real and the imaginary part.

Bogdanova et al. also reached the same conclusion [15]. They calculated the shift ΔE, the width Γ and the yield Y for the Σ^- atomic and the Σ hypernuclear level for the p-state for the at rest hypernuclear data for ${}^{12}_{\Sigma}C$ [19, 20].

The study of Σ-atomic data is a source of information for the ΣN interaction, as was told. The Batty potential was used a lot for the study of Σ-hypernuclei, as it is next described.

Stepien-Rudzka and Wycech calculated the Σ-atomic data by using the OBE Nijmegen potentials [21]. Also Yamada used the same potentials for a combined analysis of Σ-atomic and ${}^{4}_{\Sigma}He$ data. He criticized the work of Rudzka-Wycech; they neglected the region outside the nucleus $R > R_c$ (R_c: nuclear radius) which is the main source of the strong interaction effect. In that region the contribution from the p-state of the ΣN interaction is not negligible [22].

Oset et al. [23] modified the Batty potential in order to include saturation effects and by using the two potentials they calculated the Σ atomic data and Σ hypernuclear states. Tadokoro and Akaishi [24] used the above mentioned phenomenological saturation potential together with a microscopical potential for the theoretical calculation of the in-flight spectrum of ${}^{208}_{\Sigma}Pb(K^-, \pi^+)$.

Also Hayano [25] used this potential successfully for the representation of the at rest spectrum of ${}^{12}_{\Sigma}C$. Kohno-Hausmann et al. [26, 18] studied strong interaction effects in Σ-atoms in relation with the Σ hypernuclear spectra.

The perspectives of this work is the general study of all the Σ-atomic data with a more realistic potential, like the one done by the above mentioned authors.

4. Discussion

The study of Σ-atoms is directly connected with the other related topics: exotic nuclei and exotic atoms.

(i) Combined analysis of Σ-atoms and Σ-hypernuclei

The need for a combined analysis of Σ-atoms and Σ-hypernuclei became clear from the previous analysis. The recent progress in that field has revealed some physical systems where there is a mixing of the atomic and the nuclear state; these systems are described below.

The system of Σ-atoms and Σ-hypernuclei is an example of *exotic halo* which is a system of a nucleus and a strong interacting exotic particle [27]. In the exotic halo the particle wavefunction extends a lot outside the nucleus.

In the case where a Σ^- particle is bound to a medium heavy nucleus, the strong potential coexists with an attractive Coulomb potential [28]. Then we have the formation of a *Coulomb Assisted Hybrid Bound State*: CAHBS.

The case of Σ *hyperatom* is also very interesting. A hyperatom is an atom-nucleus hybrid bound state. Representative example is the ${}^{9}_{\Sigma}Be$ [29] (there is also a Ξ hyperatom: ${}^{12}_{\Xi}C$ [29]).

These states are of hybrid character and can be called either *excited* Σ-*hypernuclei*, or *deeply bound* Σ-*atoms* [27, 12].

(ii) Other exotic atoms and exotic nuclei

The production and study of other exotic atoms, like the Kaonic-atoms, the antiprotonic-atoms and the pionic-atoms is very similar with that of Σ-hypernuclei [8].

The K^--atoms were experimentally studied together with the Σ-atoms and they were studied theoretically with the same optical potential [2, 8, 9, 12].

The $\hat{p}$-atoms initially were studied together with the Σ^--atoms and the K^--atoms [9]. But since 1985 there were new experiments at CERN [4]. The measurement of the strong interaction spin-orbit effects in $\hat{p}-{}^{174}Yb$ is very interesting [4].

The most interesting are the deeply bound pionic atoms. They are deeply bound pionic states of hybrid character, between pionic nuclei and pionic atoms that form a pion halo around the nucleus [27]. The production of pionic atoms via the (d, 2p) interaction was applied successfully in SATURNE for the production of the deeply bound $\pi^- - {}^{208}Pb$ atom [27, 30].

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