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WKB EQUIVALENT POTENTIALS FOR THE q-DEFORMED HARMONIC AND ANHARMONIC OSCILLATORS *

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Abstract

WKB equivalent potentials (WKB–EP) giving the same WKB spectrum as the q-deformed harmonic oscillator with symmetry $SU_q(2)$ are determined. The WKB–EP's of the anharmonic oscillator systems with $U_q(2)$ or $SU_q(1,1)$ symmetries are also calculated.

1. Introduction

Quantum algebras [1, 2] have been attracting much attention recently as mathematical tools for solving the Yang–Baxter equation in quantum inverse problems and in statistical mechanics. A relevant collection of original papers is given by Jimbo [3]. The quantum algebras are one parameter special deformations of the universal enveloping algebras of complex simple Lie algebras corresponding to the trigonometric solutions of the quantum Yang–Baxter equation. These algebras are also concrete examples of Hopf algebras [4, 5]. The quantum algebras are in duality with the *quantum groups* or *quantum group matrices* which are related to the transformations of the Manin [6]–Woronowicz [7] *quantum plane.* A recent review of these issues is given by Dobrev [8]. The deformed $SU_q(2)$ algebra can be realised by using as basic algebraic structure the *q-deformed harmonic oscillator*

* Presented by C. Daskaloyannis

introduced by Biedenharn [9] and Macfarlane [10]. Assuming that the basic formulae of the oscillator representations of the algebra generators are valid after deformation, the q-deformed versions of the classical Lie algebras are generated, as $U_q(N)$, $SU_q(1,1)$ etc., and many of these algebras are actually the subject of intensive investigation effort.

The q-deformed harmonic oscillator [9, 10] is determined by the creation and annihilation operators a^+ and a, which satisfy the relations:

$$aa^+ - q^{-1}a^+a = q^N, \quad [N, a^+] = a^+, \quad [N, a] = -a, \tag{1}$$

where N is the *number* operator. The relevant Fock space is defined by the eigenvectors:

$$a|0>= 0, \qquad |n>= \frac{(a^+)^n}{([n]!)^{1/2}}|0>,$$

where the q-factorial is defined as

$$[n]! = [n][n-1]\dots[1],$$

and

$$[x] = \frac{\sinh(\tau x)}{\sinh(\tau)}, \qquad \tau = \ln q. \tag{2}$$

The Hamiltonian of the q-deformed oscillator is

$$H = \frac{\hbar\omega}{2}(aa^+ + a^+a),$$

with eigenvalues:

$$E(n) = \frac{\hbar\omega}{2}([n] + [n+1]).$$

Using two q-deformed algebras with generators a_1, a_1^+, N_1, and a_2, a_2^+, N_2, the q-deformed $SU_q(2)$ [9, 10] is defined:

$$J_+ = a_1^+a_2, \quad J_- = a_2^+a_1, \quad J_0 = \frac{1}{2}(N_1 - N_2). \tag{3}$$

The operators defined by eqs. (3) satisfy the commutation relations:

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = [2J_0],$$

where [.] is defined by eq. (2).

The q-deformed version of the $SU_q(1,1)$ is defined [11, 12] by:

$$K_+ = a_1^+ a_2^+, \quad K_- = a_1 a_2, \quad K_0 = \frac{1}{2}(N_1 + N_2 + 1).$$

The above generators satisfy the commutation relations:

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -[2K_0].$$

2. Construction of the WKB–EP

Following techniques of the inverse scattering method [13], we can calculate different potentials $V(x)$ having an energy spectrum exactly the same as the one described by the function:

$$E_n = f(n), \quad \text{orbytheinversefunction} \quad n = n(E). \tag{4}$$

These potentials are in the same equivalence class, corresponding to the same energy spectrum. The family of the equivalent potentials can be restricted by imposing restrictions on their shape, as symmetricity around the zero, annihilating far from zero, smoothness etc. The constructions of the exact potential do not give always simple shapes of the potential nor one unique potential form. A simpler idea was to consider a smooth symmetric potential, whose the WKB energy spectrum is the same as in eq. (4). We shall call this potential WKB–Equivalent Potential (WKB–EP). This WKB–EP should give us an idea how the exact potential is, and useful properties as the potential range or the mean square radius of the potential can be approximately estimated.

Let us consider a symmetric potential $V(x)$, with an absolute minimum at $x = 0$, as it is shown in Fig. 1. The first order WKB approximation of the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

defines the WKB energy spectrum

$$(n + \frac{1}{2})\pi = 2\left(\frac{2m}{\hbar^2}\right)^{1/2} \int_0^x (E - V(x'))^{1/2} dx',$$

where x is the turning point satisfying the relation

$$E = V(x) \text{ and } E_{min} = V_{min} = V(0), \quad n(E_{min}) + \frac{1}{2} = 0.$$

Wheeler [14]. reported by Chadan and Sabatier [13], has proposed a simple method to construct the WKB-EP to some definite energy spectrum as in eq. 4. This method can be summarized as follows:

Let us define the "*inclusion*"

$$I(E) = 2\int_0^x (E - V(x'))dx'.$$

After using the inverse Abel transform the following equation can be shown:

$$I(E) = \left(\frac{\hbar^2}{2m}\right)^{1/2} \int_{E_{min}}^{E} (E - E')^{-1/2}\left(n(E') + \frac{1}{2}\right) dE'.$$

From the inclusion we can define the "*excursion*":

$$X(E) = \frac{\partial I}{\partial E} = 2x,$$

and from the relation $E = V(x)$ we can find the potential $V(x)$ by inversion of the equation:

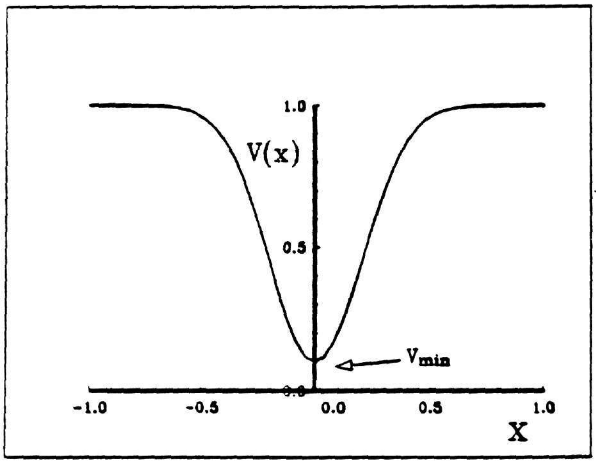

Figure 1: General form of a symmetric potential

$$X(V) = 2x \quad \rightarrow \quad V = V(x).$$

Using this method we can construct WKB–EP's for any energy spectrum if the functional form (4) of the sequence of the energy levels is known. These WKB–EP's are not the exact ones but have similarities to the exact ones. In the case of systems having symmetries defined by the q–deformed algebras, the functional forms (4) of the energy spectra are proportional to the eigenvalues of the corresponding Casimir operators, i.e. we have examples of exactly solvable problems [15]. The corresponding potentials can then be constructed.

3. Potential of the q-deformed harmonic oscillator

In the case of the q-deformed harmonic oscillator we distinguish two physically interesting cases: in the first one q is a complex number with $|q| = 1$ (i.e. $q = e^{i\tau}$, τ is a real phase). while in the second case the q–number is a real one (i.e. $q = e^{\tau}$, τ is real).

3.1 Case of an imaginary phase $q = e^{i\tau}$

The *q-oscillator* has a spectrum given by

$$E_n = \frac{\hbar\omega}{2}\frac{\sin(\tau(n+1/2))}{\sin(\tau/2)},$$

$\tau = 2\pi k/l$, k and l are natural numbers

with $0 \leq k < l$ and the g.c.d.$(k,l) = 1$.

In this case, the energy eigenvalues have discrete values:

$$| E_n | \leq E_{max} = \frac{\hbar\omega}{2}\frac{1}{|sin(\tau/2)|},$$

where the values of E_n are bounded:

$$0 \leq E_n \leq E_{max}, \quad E_{min} = \min E_n = 0.$$

The inversion of the spectrum gives:

$$n(E) + \frac{1}{2} = \frac{1}{\tau}\arcsin\left(\frac{2E}{\hbar\omega}\sin(\tau/2)\right).$$

In ref. [16] it is shown that:

$$x = \frac{1}{\tau}\left(\frac{\hbar^2}{2m}\right)^{1/2}\left(\frac{2\sin(\tau/2)}{\hbar\omega}\right)^{1/2}\sqrt{2}F(\delta,m), \tag{5}$$

where

$$\delta = \arcsin\left(\sqrt{\frac{2s}{s+1}}\right), \qquad m = \sqrt{\frac{s+1}{2}}$$

and

$$s = \frac{2V\sin(\tau/2)}{\hbar\omega}.$$

The function $F(\delta, m)$ is the elliptic integral [17]. After integration by parts, eq. (5) can be written as follows:

$$x = 2s^{1/2}R_e\left[1 - \int_0^s \frac{(s-t)^{1/2}tdt}{(1-t^2)^{3/2}}\right], \tag{6}$$

where

$$R_e = \frac{1}{\tau}\left(\frac{\hbar^2}{2m}\right)^{1/2}\left(\frac{2\sin(\tau/2)}{\hbar\omega}\right)^{1/2}.$$

By further Taylor expanding the integral in the right hand side of eq. (6) we reduce it to a series form

$$x = 2s^{1/2}R_e\left[1 + \sum_{n=1}^{\infty} s^{2n}I_{2n-1}\right], \tag{7}$$

where

$$I_n = \int_0^1 (1-u)^{1/2}u^n du = B(n+1, 3/2)$$

and $B(x,y)$ is the usual Beta function [17]. If we square eq. (7) we take

$$\left(\frac{x}{2R_e}\right)^2 = s\left(1 + \frac{4}{15}s^2 + \frac{16}{105}s^4 + \frac{320}{3003}s^6 + \ldots\right).$$

This series can be inverted to give

$$s = \frac{x^2}{2R_e^2} - \frac{8}{15}\left(\frac{x^2}{2R_e^2}\right)^3 + \frac{4448}{1575}\left(\frac{x^2}{2R_e^2}\right)^5 - \frac{345344}{675675}\left(\frac{x^2}{2R_e^2}\right)^7 + \ldots$$

Thus finally

$$V(x) = \left(\frac{\tau}{2\sin(\tau/2)}\right)^2 \frac{m\omega^2}{2}x^2$$

$$\left[1 - \frac{8}{15}\left(\frac{x}{2R_e}\right)^4 + \frac{4448}{1575}\left(\frac{x}{2R_e}\right)^8 - \frac{345344}{675675}\left(\frac{x}{2R_e}\right)^{12} + \ldots\right]. \tag{8}$$

The form of the potential V(x) of eq. (8) is shown in fig. 2 for various values of the parameter τ. For $x \to \infty$ the potential goes to a finite limiting value. These potentials have similarities to the Pöschl–Teller potential [18]. The Pöschl–Teller potential has similar shape with the Woods–Saxon potential, which is widely used in nuclear physics [19], but has a smooth derivative at the origin. The Pöschl–Teller potential has been recently used in the hypernuclear physics [20] as an alternative to the Woods–Saxon one.

3.2 Case of a real phase $q = e^{\tau}$

The *q-oscillator* has a spectrum given by

$$E_n = \frac{\hbar\omega}{2}\frac{\sinh(\tau(n+1/2))}{\sinh(\tau/2)} \tag{9}$$

and

$$n(E) + \frac{1}{2} = \frac{1}{\tau}\sinh^{-1}\left(\frac{2E}{\hbar\omega}\sinh(\tau/2)\right).$$

The Wheeler theory gives [16]:

$$x = \left(\frac{2}{m\omega^2}\right)^{1/2}\frac{2\sinh(\tau/2)}{\tau}V^{1/2}\ {}_3F_2(1/2,1/2,1;3/4,5/4,-s^2), \tag{10}$$

where

$$s = \frac{2V\sinh(\tau/2)}{\hbar\omega}.$$

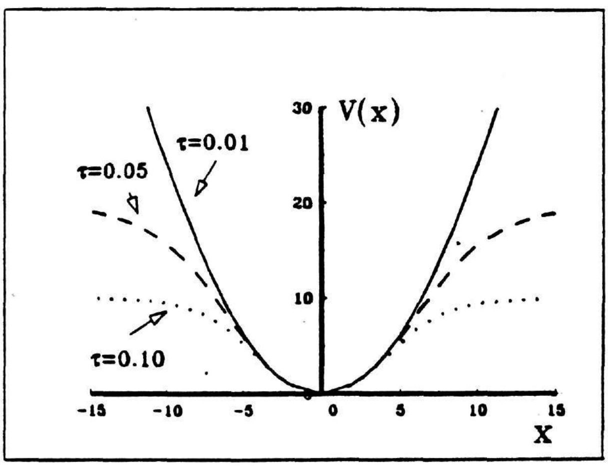

Figure 2: WKB–EP V(x) for the q–deformed oscillator, with $q = e^{i\tau}$

In the same way if we square eq. (10) and take the first few terms of the generalized hypergeometric function we do have that

$$x^2 = R_h^2 E\left[{}_3F_2(1/2, 1/2, 1; 3/4, 5/4, -s^2)\right]^2,$$

where

$$R_h = \frac{1}{\tau}\left(\frac{\hbar^2}{2m}\right)^{1/2}\left(\frac{2\sinh(\tau/2)}{\hbar\omega}\right)^{1/2}.$$

Using a similar method as in the case of a real phase we conclude that

$$V(x) = \left(\frac{\tau}{2\sinh(\tau/2)}\right)^2 \frac{m\omega^2}{2}x^2$$

$$\left[1 + \frac{8}{15}\left(\frac{x}{2R_h}\right)^4 + \frac{4448}{1575}\left(\frac{x}{2R_h}\right)^8 + \frac{345344}{675675}\left(\frac{x}{2R_h}\right)^{12} + \dots\right],$$

The shape of the potential $V(x)$ is shown in fig. 3. The WKB–EP in this case goes to infinity when $x \to \infty$ and for $\tau = 0$ the potential is degenerated to the classical harmonic oscillator potential. We shall mention here that the q-deformed oscillator near the origin behaves like a *sextic* anharmonic oscillator:

$$V(x) \sim x^2 + cx^6 + \dots$$

Therefore in second order approximation for large values of the *range* R_h the Taylor expansion over τ of energy spectrum (9) should represent fairly well the spectrum of the sextic anharmonic oscillator without the x^4 term.

4. Anharmonic oscillator having an $SU_q(1,1)$ symmetry

The energy spectrum of a system with $SU_q(1,1)$ symmetry is given by:

$$E_n = E_0 - A\frac{\sin(\tau(n - N/2))\sin(\tau(n + 1 - N/2))}{\sin^2(\tau)}.$$

For $\tau \to 0$ this corresponds to a *Morse* or *Pöschl-Teller* spectrum. The Wheeler theory gives [21]:

$$x = \frac{1}{2}\left(\frac{\hbar^2}{2m}\right)^{1/2}\left(\frac{2\sin^2(\tau)}{A}\right)^{1/2}\sqrt{2}F(\delta, m),$$

where

$$m^2 = \sin^2(\tau)\frac{V - V_{min}}{A} + \cos(\tau N/2),$$

and

$$\delta = \arcsin\left(\frac{1}{m}\frac{\sin\tau}{\sin(N\tau/2)}\sqrt{\frac{V-V_{min}}{A}}\right).$$

The potentials in this case can be taken by inversion of the function

$$\frac{x}{R} = \int\limits_{z_{min}}^{z} \frac{dt}{(z-t)^{1/2}(1-t^2)^{1/2}}, \tag{11}$$

with

$$R = \frac{1}{2\tau}\left(\frac{\hbar^2}{2m}\right)^{1/2}\left(\frac{2\sin\tau}{A}\right)^{1/2},$$

where for the $SU_q(1,1)$ spectrum z and z_{min} take the values

$$z = 2\sin^2\tau\frac{E-E_{min}}{A} + \cos(N\tau), \tag{12}$$

and

$$z_{min} = \cos(N\tau). \tag{13}$$

After expanding the $(1-t^2)^{1/2}$ in the right hand side of eq. (11), we find:

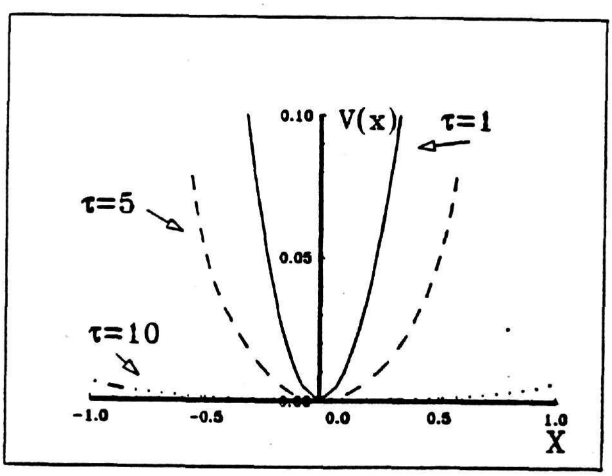

Figure 3: WKB–EP V(x) for the q–deformed oscillator, with $q = e^{\tau}$

$$\frac{x}{R} = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!\Gamma(1/2)} \int_{z_{min}}^{z} \frac{t^{2n}}{(z-t)^{1/2}} dt.$$

The integrals in the above equation can be represented as follows:

$$\int_{z_{min}}^{z} \frac{t^{2n}}{(z-t)^{1/2}} dt = z^{2n+1/2} B_{1-z_{min}/z}(1/2, 2n+1)$$

$$= (z - z_{min})^{1/2} z_{min}^{2n} \, {}_2F_1(-2n, 1; 3/2; -(z - z_{min})/z_{min}),$$

where $B_u(x, y)$ is the incomplete Beta function [17].

If we Taylor expand the right hand side of eq. (11), we bring it into the form

$$\left(\frac{x}{2R'}\right) = y^{1/2} \left[1 + A_1 y + A_2 y^2 + \ldots\right],$$

where

$$R' = \left(1 - z_{min}^2\right)^{-1/2} R,$$

$$y = z - z_{min},$$

and the first few of the coefficients A_i are calculated as follows:

$$A_1 = \frac{2}{3} \frac{z_{min}}{1 - z_{min}^2},$$

$$A_2 = \frac{2^2}{3 \cdot 5} \left[3 \frac{z_{min}^2}{(1 - z_{min}^2)^2} + \frac{1}{1 - z_{min}^2}\right],$$

$$A_3 = \frac{2^3}{3 \cdot 5 \cdot 7} \left[3 \cdot 5 \frac{z_{min}^3}{(1 - z_{min}^2)^3} + 3^2 \frac{z_{min}}{(1 - z_{min}^2)^2}\right],$$

$$A_4 = \frac{2^4}{3 \cdot 5 \cdot 7 \cdot 9} \left[3 \cdot 5 \cdot 7 \frac{z_{min}^4}{(1 - z_{min}^2)^4} + 3^2 \cdot 5 \cdot 2 \frac{z_{min}^2}{(1 - z_{min}^2)^3} + 3 \cdot 2^2 \frac{1}{(1 - z_{min}^2)^2}\right].$$

By inversion of the previous series we find the following series

$$z - z_{min} = a_0 \left(\frac{x}{2R'}\right)^2 + a_1 \left(\frac{x}{2R'}\right)^4 + a_2 \left(\frac{x}{2R'}\right)^6 + \ldots,$$

where

$$a_0 = 1,$$

$$a_1 = -\frac{4}{3}\frac{z_{min}}{1-z_{min}^2},$$

$$a_2 = \frac{4}{45}\frac{23z_{min}^2-6}{(1-z_{min}^2)^2},$$

$$a_3 = -\frac{16}{315}\frac{67z_{min}^3-36z_{min}}{(1-z_{min}^2)^3}.$$

Replacing z and z_{min} by eqs. (12) and (13) finally we get the following series expansion for the potential:

$$V(x) = V_{min} + \frac{A}{4}\left(\frac{\tau\sin(N\tau)}{\sin^2\tau}\right)^2 u^2$$

$$\left[1 - \frac{2}{3}\frac{\tau^2\cos(N\tau)}{\sin^2\tau}u^2 + \frac{1}{45}\frac{\tau^4(23\cos^2(N\tau)-6)}{\sin^4\tau}u^4\right.$$

$$\left. - \frac{2}{315}\frac{\tau^6(67\cos^2(N\tau)-36)\cos(N\tau)}{\sin^6\tau}u^6 + \cdots\right], \tag{15}$$

where

$$u = \frac{\sqrt{2mA}x}{\hbar}.$$

The above formula seems to be very complicated, but it can be easily tested. In [21] it was shown that for $\tau \to 0$ the above potential should be the well known modified Pöschl–Teller potential:

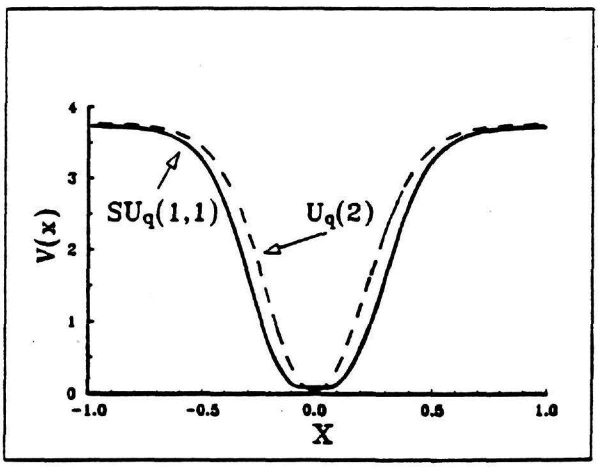

Figure 4: WKB–EP for systems having $SU_q(1,1)$ symmetry and $U_q(2)$ symmetry (dashed line)

$$V(x) = V_{min} + \frac{AN^2}{4}\tanh^2\left(\frac{\sqrt{2mA}x}{\hbar}\right).$$

It is trivial to see that the series expansion given by eq. (15) is indeed the power series expansion of the above function for $\tau \to 0$. This means that the WKB–EP of a system with the deformed $SU_q(1,1)$ symmetry is a deformation of the Pöschl–Teller potential. It is well known [15, 24] that the Morse potential has an energy spectrum proportional to the Casimir eigenvalues of $SU(1,1)$. In [21] it is shown that the WKB–EP of the Morse potential is the Pöschl–Teller potential. Therefore the potential given by eq. (15) is indeed the deformed version of the Pöschl–Teller potential. The deformed $SU_q(1,1)$ symmetry has been applied recently to the study of diatomic molecules [22].

In fig. 4 the form of the WKB–EP is shown.

5. Anharmonic oscillator with an $U_q(2) \supset O(2)$ symmetry

The energy spectrum of a system with $U_q(2)$ symmetry is given by:

$$E_n = E_0 + A\frac{\sin(2\tau n))\sin(2\tau(N-n))}{\sin^2(\tau)}.$$

For $\tau \to 0$ it corresponds to a Morse or Pöschl-Teller spectrum. The Wheeler theory gives:

$$x = \frac{1}{2}\left(\frac{\hbar^2}{2m}\right)^{1/2}\left(\frac{2\sin^2(\tau)}{A}\right)^{1/2}\sqrt{2}F(\delta,m),$$

where

$$m^2 = \sin^2(\tau)\frac{V-V_{min}}{A} + \cos(\tau(N+1)/2),$$

and

$$\delta = \arcsin\left(\frac{1}{m}\frac{\sin\tau}{\sin((N+1)\tau)}\sqrt{\frac{V-V_{min}}{A}}\right).$$

For the $U_q(2)$ spectrum z and z_{min} take the values

$$z = 2\sin^2\tau\frac{E-E_{min}}{A} + \cos[2\tau(N+1)], \tag{16}$$

and

$$z_{min} = \cos[2\tau(N+1)].$$

For this potential eq. (14) is valid, but the values of z and z_{min} given by eqs. (16) and (17) should be used. The deformed $U_q(2) \supset O(2)$ symmetry has been applied to the study of diatomic molecules [23]. The form of the WKB–EP is shown in fig. 4.

6. Conclusions

The WKB-EP's for the q–deformed harmonic and anharmonic oscillators are analytically calculated. The potential representing the q–oscillator spectrum, q being a phase, behaves like a modified Pöschl–Teller potential. Although if q is a real number the corresponding potential tends to infinity as x goes to zero, for q near to unity the potential can be approximated by a sextic anharmonic oscillator. The potentials of systems with $SU_q(1,1)$ or $U_q(2) \supset O(2)$ symmetries are generated by a deformation of the modified Pöschl-Teller potential.

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