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AN APPLICATION OF THE DEMOCRATIC MAPPING IN REALISTIC SYSTEMS *

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Abstract

The democratic mapping is used for the calculation of low lying states of nuclei in the sd and fp shells. In addition to demonstrating the applicability of the method in realistic cases where many non-degenerate levels are present, the method allows for the ranking of the various bosons according to their importance as building blocks of low lying states. It is *proven* that the s and d bosons are the most important building blocks, followed by the d' and g bosons. Thus one of the basic *assumptions* of the Interacting Boson Model (IBM) is *proven* to be correct. Very good agreement between the boson calculation and the shell model results is obtained for $A = 20$ nuclei when 12 bosons are taken into account, while an even larger number of bosons is required to reproduce the low-lying states of the $A = 44$ nuclei. In order to obtain equally good results with a smaller number of bosons one needs to introduce effective boson hamiltonians which correspond to truncated fermion spaces.

1. Introduction

Since the introduction of the Interacting Boson Model (IBM) by Arima and Iachello [1,2] (for recent overviews see [3,4]), many attempts have been made to establish a connection between this model and the shell model. A necessary step in this boson mapping

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process (see [5] and references therein) is the transition from the fermion space of the shell model to the collective boson subspace. Since the number of bosons describing a particular collective nucleus in the IBM framework is constant, only number conserving boson mappings are suitable for this purpose. An approach widely used is the Otsuka–Arima–Iachello (OAI) mapping method [6]. It is a seniority-based state mapping, that is, the seniority [7] classification of shell-model states is carried over onto a similar classification of boson states. The boson images of fermion operators are determined by computing matrix elements between shell-model states with good seniority. It is required that the matrix elements of the boson images of the various operators in the boson basis be equal to the matrix elements of the corresponding fermion operators in the fermion basis. Therefore the OAI mapping is a mapping of the Marumori type [8]. A different approach, based on the requirement that the boson images of the various pair and multipole operators satisfy the same commutation relations (i.e. a mapping of the Belyaev–Zelevinsky–Marshalek (BZM) type [9,10]) is the Bonatsos, Klein and Li (BKL) method [11–13].

In a recent paper [14], to be hereafter referred to as I, an alternative mapping method was proposed, which is, as the OAI method, of the Marumori type [8]. This new method is, however, different from the OAI approach, since it treats on equal footing all shell-model states which are mapped onto corresponding boson states, i.e without the hierarchy implicitly assumed in the OAI approach. This is the reason why the method was named in I "democratic".

So far, the democratic method was only tested in the $f7/2$ shell. This test proved succesful since the energy spectra obtained with this method for the $A = 45 - 48$ nuclei were found [14] to be in satisfactory agreement with the shell-model results. However, as claimed in I, the real advantage of this new mapping method is that it can easily be applied to realistic shell-model spaces where many single-particle orbitals are involved. This efficiency of the method is due to the flexible way in which the necessary coefficients of fractional parentage are calculated and stored for further use [15,16]. Such a realistic application is discussed in the present paper where the democratic mapping method is applied to the sd and fp shells.

It is customary in IBM calculations to consider only s and d bosons for the description of the basic features of the low-lying spectra of collective nuclei [1–4]. However, the boson space formed by s and d bosons only is too small and in order to give a better description of nuclear spectra it has been found necessary to include additional bosons, like the g boson [17-19] and s' and d' bosons [20], as well as proton-neutron bosons with $T = 1$ [21] and $T = 0$ [22]. The s' and d' bosons are important for the description of intruder states [23], while the proton-neutron bosons with $T = 1$ and $T = 0$ have been found particularly useful in applications in the sd shell [24,25], where the valence protons and neutrons occupy the same major shell.

From the above, it becomes evident that to reproduce by a mapping procedure results obtained in large shell-model spaces it is necessary to consider boson spaces of correspondingly large dimensions. In the case of the OAI mapping the g boson has been recently introduced [26], while in the BKL method it appears as a consistency requirement in the next-to-lowest order approximation [11]. s' and d' bosons have also been considered in the BKL method, along with other bosons, like f and p bosons of negative parity [12], which are useful for the description of low-lying octupole states [27]. Several non-degenerate levels have been considered in both the OAI [28] and the BKL [12] approaches.

The selection of the bosons to be included in the mapping procedure has been based so far either on experience, as in the OAI case, or on mathematical consistency requirements, as in the BKL case. The flexibility of the democratic approach allows for a different kind of test to be performed. One can start with a relatively large number of bosons. In the case of the sd shell, for example, one can consider out of the 28 possible bosons the 12 bosons which lie lowest in energy. It is clear that with such a rich space one can reproduce the low-lying shell model results quite accurately. One can then perform 12 different calculations, each involving 11 of the bosons previously considered and decide to permanently remove the boson which causes the least damage to the agreement with the shell model results. Then one is left with 11 bosons and can perform 11 different 10-boson calculations, in order to decide which of the 11 bosons now in hand is the less important. Continuing this procedure, one can rank the bosons, in an impartial way, according to their importance. The results of this investigation for the $A = 20$ nuclei are discussed in sect. 2.

The success of the IBM lies in the fact that with relatively few degrees of freedom one can account for many of the properties of the low-lying spectra of nuclei. To accomplish such a result one needs to consider an effective boson hamiltonian. The manner in which such an effective hamiltonian can be obtained in the framework of the democratic mapping is discussed in sect. 3, where the method is applied to the sd shell. In sect. 4 a similar calculation for the fp shell is performed, while sect. 5 contains the conclusions of this work.

2. Relative importance of bosons in the sd shell

In this section we apply the democratic mapping method, described in [14], to the sd shell. Our aim is to reproduce in the framework of IBM the shell model results for the low-lying spectra of the $A = 20$ nuclei. These nuclei have four valence fermions outside the ^{16}O closed core and the full fermion space consists of 640 antisymmetric states. The number of states for each set of (J, T) values is shown in table 1. In this table N denotes the total number of states for a given (J, T) set of values, while n the number of low-lying states the energies of which we are interested in reproducing with our mapping method.

In our shell-model calculation in the sd shell we have assumed full configuration mixing and placed the valence fermions in the $0d5/2$, $1s1/2$ and $0d3/2$ orbitals of a harmonic oscillator, with $\hbar\omega = 14.4$ MeV. For the one-body part of the fermion hamiltonian we have used the experimental single-particle energies [29] of ^{17}O :

$$\epsilon_{5/2} = 0, \quad \epsilon_{1/2} = 0.87\text{MeV}, \quad \epsilon_{3/2} = 5.08\text{MeV}, \tag{1}$$

while for the two-body part the matrix elements of Preedom and Wildenthal [30].

In the sd shell one can form 14 fermion pairs with $T = 1$ and another 14 pairs with $T = 0$. Therefore, in the manner of ref. [14], one can associate 28 bosons with two-fermion eigenstates. Including all 28 bosons is technically difficult and, moreover, will most certainly result in the linear dependence of the four-fermion states which will be associated with bosons in the manner of ref. [14]. Since we are interested in explaining the low-energy spectra of the $A = 20$ nuclei only, it is reasonable to assume that the bosons lying lowest in energy should be the most important ones. We have decided, therefore, to associate with bosons the 12 lowest two-fermion eigenstates. The 12 bosons selected from this procedure are listed in the first line of table 2. Following the usual notation, we use in table 2 and elsewhere the symbols s, d, g for $T = 1$ bosons with $J = 0$, $J = 2$ and $J = 4$, correspondingly. For $T = 0$ bosons we use the notation [31] θ_J. Unprimed bosons are the ones lying lowest in energy, while primed bosons are the next lowest lying ones.

Table 1

Number of four-fermion antisymmetric states in the sd shell

J	T	N	n	J	T	N	n	J	T	N	n
0	0	21	3	0	1	16	1	0	2	9	2
1	0	31	1	1	1	54	3	1	2	12	1
2	0	36	5	2	1	66	4	2	2	21	3
3	0	45	2	3	1	69	5	3	2	21	1
4	0	44	5	4	1	50	3	4	2	15	2
5	0	24	1	5	1	34	1	5	2	6	1
6	0	17	2	6	1	16	1	6	2	3	1
7	0	5	1	7	1	7	1				
8	0	3	1	8	1	1	1				

Table 2

Boson spaces considered in the calculation in the sd shell

M_1	s	d	g	s'	d'	θ_1	θ_2	θ_3	θ_4	θ_5	θ'_1	θ'_3
M_2	s	d	g	s'	d'	θ_1	θ_2	θ_3	θ_4	θ_5		θ'_3
M_3	s	d	g	s'	d'	θ_1	θ_2	θ_3	θ_4	θ_5		
M_4	s	d	g	s'	d'	θ_1	θ_2	θ_3		θ_5		
M_5	s	d	g	s'	d'	θ_1		θ_3		θ_5		
M_6	s	d	g	s'	d'			θ_3		θ_5		
M_7	s	d	g		d'			θ_3		θ_5		
M_8	s	d	g		d'					θ_5		
M_9	s	d	g		d'							
M_{10}	s	d			d'							
M_{11}	s	d										

Table 3

Overall quality of results obtained by the various boson models in the sd shell

		N	σ	N'	σ'
M_1		52	0.427	52	0.427
M_2	θ'_1	52	0.526	50	0.443
M_3	θ'_3	52	0.619	50	0.529
M_4	θ_4	52	0.780	49	0.611
M_5	θ_2	52	1.264	49	0.769
M_6	θ_1	52	1.482	47	0.984
M_7	s'	52	1.831	47	1.251
M_8	θ_3	51	2.171	46	1.598
M_9	θ_5	50	2.465	45	1.946
M_{10}	g	33	3.609	30	3.124
M_{11}	d'	13	4.566	13	4.566

After the selection of the bosons, discussed above, the matrix elements of the boson hamiltonian were determined following the procedure described in ref. [14]. Consequently we were able to obtain energy spectra for the $A = 20$ nuclei by diagonalizing the boson hamiltonian in the basis of two-boson vectors. From the results of this IBM calculation, which we shall call henceforth M_1 calculation, we obtain a very good description of the 52 low lying states of table 1. The quality of the fit can be seen in table 3, which shows the rms deviation σ. This is defined as

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(E_i(SM) - E_i(b))^2}, \tag{2}$$

where N is the number of states included in the calculation. $E_i(SM)$ are the energies resulting from the shell-model calculation and $E_i(b)$ are the corresponding energies resulting from the boson calculation.

As table 3 shows, the results obtained with the 12 bosons included in the M_1 model are in very good agreement with the shell-model predictions. However, the large number of bosons included in the M_1 calculation makes difficult the application of such a model to heavier nuclei in the sd shell. We have examined, therefore, the possibility of reducing the number of bosons included in the model, making sure in parallel that we cause the least possible damage to the agreement between the results of the boson and shell-model calculations. This classification of the bosons according to their importance in the low-lying spectra of the $A = 20$ nuclei has been obtained in the following manner. We have attempted 12 different calculations, each of them involving 11 of the bosons previously used. In each of these calculations we have followed exactly the same procedure with that adopted in the M_1 calculation, i.e the democratic method was applied to produce the appropriate hamiltonian which, in turn, was diagonalized to produce the energy spectrum. Finally, for each set of results the rms deviation, defined in eq. (2), was computed. It turns out that the smallest σ is obtained when θ_1' is removed. We conclude, therefore, that θ_1' is the least important of the 12 bosons used in M_1. The best 11-boson model we can have is then the model which contains all bosons used in M_1 except θ_1'. This new model we call henceforth M_2. The bosons used in M_2 are shown in table 2, while the corresponding σ

is shown in table 3. It should be remarked at this point that the removal of the σ_1' boson mainly affects two of the states of table 1, namely the first $J = 1$, $T = 0$ and the second $J = 3$, $T = 0$, both of which lie relatively high in the fermion spectrum. Most of the increase of σ from 0.427 for M_1 to 0.526 in M_2 is due to these two states. It is reasonable then to exclude such "pathological" states from our procedure. When excluding these two states we see that σ for M_2 is just 0.443, only slightly higher than the 0.427 value found for M_1.

We can now continue this classification of boson states by starting from the 11 bosons of M_2 and performing 11 different 10-boson calculations. The least useful boson in this case is found to be θ_3'. Therefore M_3 includes all bosons of M_2 except θ_3'. Continuing in this way we find, as shown in tables 2 and 3, that the next bosons to be removed are θ_4, θ_2, θ_1, s', θ_3 and θ_5. One is then left with M_9, which contains only 4 bosons, the s, d, d' and g ones. We remark that by now all the $T = 0$ bosons have been removed. The boson space is, however, still quite rich, so that 50 of the original 52 states of table 1 can be accounted for and, as table 3 shows, the energies of 45 of the accounted states are in satisfactory agreement with the shell-model values. The next "victim" is g, followed by d'. In the last two steps, as seen in table 3, one can account with the remaining bosons only a small fraction of the original 52 states of table 1. One is finally left with M_{11}, which contains only the last two bosons, which turn out to be the most important building blocks of the low lying states, namely the s and d bosons. The bosons present in each step are shown in table 2, while the quality of the results obtained with each model is shown in table 3. Throughout this procedure particular states were giving large contributions to σ. Therefore in table 3 we give two cases: by N and σ we describe the full number of states obtained with the bosons in hand as well as the corresponding rms deviation, while by N' we indicate the number of "non-pathological" states and by σ' the rms deviation corresponding to these N' states. We remark that while N' is only slightly smaller than N, σ' is always significantly smaller than σ.

A more detailed presentation of the results obtained from the calculations, $M_1 \ldots M_{11}$, described above, is given in table 4. Table 4 lists the energy eigenvalues obtained from the IBM calculations and compares them with the shell-model results. To avoid making table

4 too lengthy we only list the energies of about half of the states which were used in the determination of the rms deviation. The states listed in table 4 were the lowest in energy but representatives of all possible J, T values have been included. Moreover, we restrict in table 4 the presentation of results only to those obtained from M_k ($k = 1, 3, 5, 7, 9, 10, 11$) calculations i.e we omit the results of the intermediate M_2, M_4 etc.

Table 4
Energies of the low-lying states of ^{20}Ne (in MeV)
obtained from the shell-model and the various IBM calculations

T	J_π	SM	M_1	M_3	M_5	M_7	M_9	M_{10}	M_{11}
0	0_1	-23.17	-23.12	-23.03	-22.96	-22.86	-21.29	-20.24	-17.71
	0_2	-17.64	-17.60	-17.26	-17.21	-14.17	-13.75	-13.19	-8.55
	0_3	-13.32	-13.29	-13.24	-13.22	-11.25	-10.60	-9.41	—
	2_1	-21.11	-20.81	-20.63	-20.28	-20.24	-19.26	-17.41	-15.82
	2_2	-17.35	-17.04	-16.67	-16.46	-16.05	-15.93	-15.51	-11.50
	2_3	-13.56	-13.33	-13.09	-12.94	-12.00	-11.55	-10.99	—
	3_1	-13.64	-13.33	-13.11	-12.72	-12.68	-12.57	-11.73	—
	4_1	-19.15	-18.87	-18.29	-17.66	-17.18	-16.67	-12.99	-12.96
	4_2	-14.08	-14.03	-13.85	-13.68	-13.61	-13.50	-11.07	—
	4_3	-13.52	-13.45	-13.22	-12.76	-11.94	-11.65	-9.89	—
	5_1	-13.09	-13.02	-12.86	-12.52	-12.52	-12.16	—	—
	6_1	-14.44	-14.28	-14.04	-13.39	-13.39	-13.03	—	—
1	1_1	-11.93	-11.08	-11.00	-10.94	-10.65	-9.44	-9.43	—
	1_2	-10.75	-10.35	-10.03	-10.00	-9.63	-9.11	-7.42	—
	2_1	-13.54	-12.84	-12.82	-12.40	-11.94	-11.43	-11.36	-10.58
	2_2	-11.20	-10.30	-10.08	-9.56	-9.41	-9.14	-8.32	—
	3_1	-12.35	-11.89	-11.75	-11.33	-11.23	-9.91	-9.21	-8.97
	3_2	-11.56	-10.94	-10.35	-10.12	-9.75	-9.11	-7.19	—
	4_1	-12.66	-12.09	-11.95	-11.70	-11.31	-10.70	-6.52	—
	5_1	-11.81	-11.73	-11.68	-11.58	-11.48	-9.87	—	—
2	0_1	-7.18	-7.18	-7.18	-7.18	-7.15	-7.15	-6.88	-6.82
	0_2	-2.46	-2.45	-2.45	-2.45	-1.98	-1.98	-1.96	0.61
	1_1	-2.65	-2.49	-2.49	-2.49	-2.49	-2.49	-2.49	—
	2_1	-4.85	-4.74	-4.74	-4.74	-4.73	-4.73	-4.40	-4.34
	2_2	-3.36	-3.25	-3.25	-3.25	-3.25	-3.25	-3.08	1.07
	2_3	-2.12	-2.03	-2.03	-2.03	-2.03	-2.03	-1.24	—
	4_1	-3.46	-3.39	-3.39	-3.39	-3.38	-3.38	-2.64	-2.37

One may observe from table 4 that all states are very well reproduced by the M_1 calculation (12 bosons). However, this agreement with the shell-model results deteriorates as the number of bosons decreases. The rate of deterioration is slow while the number of bosons is still large (up to about M_6) but becomes very rapid for a small number of bosons. There are some cases, mostly observed in $T = 2$ states, where there is no deterioration of the results but the various boson calculations produce the same energies. Such a behaviour occurs if the boson space is too rich and the fermion images of the boson states are not linearly independent. One indeed observes that the constancy of the results disappears as soon as the number of bosons becomes too small. The fact that, despite the richness of the boson space, the boson energies do not coincide with the shell-model results suggests that bosons other than the 12 considered in the present calculation influence the structure of these particular states.

Some of the results shown in table 4 are easy to explain. For example, the second $J = 0$, $T = 0$ state is well reproduced up to M_6 (see table 4), but in M_7 it becomes displaced by about 3 MeV. The reason for this behaviour is that in M_7 the s' boson, which was present up to that point, is omitted. Similarly, the second $J = 2$, $T = 0$ and the first $J = 4$, $T = 0$ states are displaced by about 4 MeV when the d' boson and the g bosons are removed, respectively.

The main conclusions of this section are then summarized as follows:

i) One can reproduce quite accurately the shell-model results for low-lying states of the $A = 20$ system by using out of the 28 possible bosons the 12 ones lying lowest in energy.

ii) Through a completely impartial method one can arrange these bosons in order of importance as building blocks of the low-lying states. It turns out that the s and d bosons are the most important building blocks, followed by d' and g. Notice that the fundamental role of the s and d bosons is here *proven*, not *assumed*.

iii) It is clear that certain bosons influence strongly particular states. For example, the s' boson influnces strongly the second $(J, T) = (0, 0)$ state, while the d' and g bosons affect mainly the second $(2, 0)$ and the first $(4, 0)$ states, respectively.

iv) It is clear that the agreement of the results of the boson calculation to the shell model results is reduced as the number of bosons used in the model is decreased.

3. Effective boson Hamiltonian for the sd shell

The results of the previous section indicate that to reproduce satisfactorily the low-lying states of a multi-orbital shell-model calculation one needs to consider a large boson space. However, in this section we discuss a modification to the democratic mapping method by which an IBM calculation, in a small boson space, can account satisfactorily for the low-lying shell-model states. This new approach is applied in this section to the sd shell while in the next section we discuss an application of the method to the fp shell.

As discussed in sect. 2, the four-fermion space, in the case of the sd shell, contains 640 states. It should be realised from the description of the method, given in ref. [14], that the democratic mapping considers the effects of all these states. This happens because in order to obtain matrix elements between the fermion images $|F_i\rangle$ of the boson states $|B_i\rangle$ one considers a summation over a complete set of fermion states. Thus the information passed through the mapping method from the fermion to the boson space is an average one and not the one specially needed for the reproduction of the low-lying states. (See [14] for a discussion on this subject.)

There are two ways to obtain a satisfactory agreement between shell-model and IBM results. The first, already applied in sect. 2, is to use a large boson space so that to bring the boson vectors as close as possible to one-to-one correspondence with the fermion states. The other is to apply the mapping method only to a subspace of the fermion states, namely the space formed by the low-lying eigenstates of the shell-model hamiltonian. The advantage of the first approach is that it produces a boson hamiltonian which is equally suitable for the description of all states irrespective of their position in the energy spectrum. On the other hand, with the second approach one will obtain an "effective" boson hamiltonian which will reflect the properties of the low-lying states only.

As an application of the second method we have considered a boson space consisting of the s, d, g and d' bosons (M_9 model). As shown in table 3, in this boson space one can account for 50 out of the 52 low lying states of the $A = 20$ nuclei, but with a large rms deviation of 2.465. Table 5 shows how this large rms value can be reduced by truncations in the space of the fermion eigenstates. Thus for each combination of J and T we can keep, instead of the full number of states, only the 80% of them lying lowest in energy, or, to be

Table 5
Dependence of the results obtained by the M_9 calculation on the number of fermion states included in the mapping

	1.0	0.8	0.6	0.4	0.2	0.1
N	50	50	50	50	43	39
σ	2.465	2.420	2.233	1.745	1.048	0.403

Table 6
Energies of the low-lying states of ^{20}Ne (in MeV) as a function of the number of fermion states included in the mapping

T	J_τ	1.0	0.8	0.6	0.4	0.2	0.1	SM
0	0_1	-21.29	-21.37	-21.40	-21.59	-23.17	-23.17	-23.17
	0_2	-13.75	-13.81	-14.04	-15.98	-17.64	-17.64	-17.64
	0_3	-10.60	-10.61	-10.84	-11.72	-13.32	—	-13.32
	2_1	-19.26	-19.30	-19.37	-19.57	-20.22	-21.11	-21.11
	2_2	-15.93	-15.95	-16.01	-16.61	-17.27	-17.35	-17.35
	2_3	-11.55	-11.57	-11.63	-12.24	-13.14	-13.56	-13.56
	3_1	-12.57	-12.59	-12.64	-12.97	-13.32	-13.54	-13.64
	4_1	-16.67	-16.73	-16.83	-17.04	-17.86	-19.15	-19.15
	4_2	-13.50	-13.52	-13.62	-13.74	-13.82	-14.08	-14.08
	4_3	-11.65	-11.68	-11.91	-12.48	-13.29	-13.52	-13.52
	5_1	-12.16	-12.21	-12.29	-12.57	-13.04	-13.09	-13.09
	6_1	-13.03	-13.14	-13.26	-13.76	—	—	-14.44
1	1_1	-9.44	-9.47	-9.55	-10.06	-10.82	-11.79	-11.93
	1_2	-9.11	-9.18	-9.45	-9.71	-10.25	-10.58	-10.75
	2_1	-11.43	-11.45	-11.51	-11.89	-12.27	-3.50	-13.54
	2_2	-9.14	-9.17	-9.25	-9.68	-10.53	-11.11	-11.20
	3_1	-9.91	-9.96	-10.11	-10.58	-11.35	-11.97	-12.35
	3_2	-9.11	-9.16	-9.41	-9.86	-10.31	-11.10	-11.56
	4_1	-10.70	-10.71	-10.83	-10.99	-11.41	-12.56	-12.66
	5_1	-9.87	-9.96	-10.37	-10.78	-11.11	-11.81	-11.81
2	0_1	-7.15	-7.18	-7.18	-7.18	-7.18	-7.18	-7.18
	0_2	-1.98	-2.03	-2.46	-2.46	-2.46	—	-2.46
	1_1	-2.49	-2.49	-2.50	-2.54	-2.64	-2.65	-2.65
	2_1	-4.73	-4.74	-4.77	-4.85	-4.85	-4.85	-4.85
	2_2	-3.25	-3.26	-3.27	-3.36	-3.36	-3.36	-3.36
	2_3	-2.03	-2.05	-2.06	-2.12	-2.12	—	-2.12
	4_1	-3.38	-3.39	-3.41	-3.46	-3.46	-3.46	-3.46

precise. the nearest integer to that value. In the next step we can keep only 60% of the states. again the ones lying lowest in energy, and so on. It should be emphasized that this particular way of truncating the fermion space has only been considered because of the simplicity of its description: one may consider other more elaborate schemes. The purpose of this schematic calculation is made clear by the results shown in table 5. By reducing the size of the fermion space down to 40% we remark that we can still build the 50 out of the 52 states given in table 1. but the rms deviation, although it falls from 2.465 for 100% of the space to 1.745 for 40% of the space. still remains quite sizeable. However, in the next two steps, the rms deviation is decreased quite drastically, although most of the 52 states under consideration can still be built. As seen in table 5, with 20% of the fermion space under consideration one can build 43 out of the 52 states with an rms deviation of 1.048. while with 10% of the space one can build 39 states with an rms deviation of 0.403.

The drastic improvement of the results obtained with the 4 bosons under consideration (s, d. d'. g) through the truncation of the fermion space is clearly seen in table 6, which shows the dependence of the low-lying eigenvalues of the boson hamiltonian on the number of fermion states considered in the mapping method. We remark that the results obtained from the boson calculation when only 10% of the fermion space is taken into account are very close to the shell model results.

The main conclusion of this section is: With few bosons one can build most of the low-lying states, but when the full fermion space is taken into account, the agreement between the results of the boson calculation and the shell model calculation is poor. However, one can drastically improve this agreement by appropriately truncating the fermion space, keeping only the few lowest lying states for each combination of J and T.

4. The fp shell

In sect. 2 we used the results of a shell model calculation in the sd shell to order the bosons according to their importance in the description of the low-lying states of the $A = 20$ nuclei. A repetition of this procedure to the fp shell, although straightforward, is very tedious due to the large dimensions of the shell-model matrices. Thus, as shown in table 7, there are altogether 4000 four-fermion states in the fp shell compared to the 640 states encountered in the sd shell. Furthermore, in the fp shell one can have 60 different fermion pairs. and thus 60 different bosons (30 $T = 1$ and 30 $T = 0$ bosons), in comparison to the 28 bosons present in the sd shell. To reproduce, therefore, the low-lying states of the $A = 44$ nuclei using the procedure of sect. 2 one needs to consider a larger number of bosons than the 12 considered in the sd case. Thus to account satisfactorily for the energies of the lowest 3 of the $(J, T) = (0, 0)$ states of ^{44}Ti we found necessary to consider a boson space consisting of 17 bosons. Obviously, it is very difficult to study the other (J, T) states of the $A = 44$ nuclei in such a large boson space and, therefore, in the following we report only results obtained by simpler models and using the procedure of sect. 3.

In our shell-model calculation in the fp shell we have assumed full configuration mixing and placed the valence fermions in the $0f7/2$, $1p3/2$, $1p1/2$ and $0f5/2$ orbitals of the harmonic oscillator potential. The energy matrices have been constructed using the renormalized two-body matrix elements of Kuo and Brown [32] together with the following empirical set of single particle energies:

Table 7

Number of four-fermion antisymmetric states in the fp shell

J	T	N	J	T	N	J	T	N
0	0	66	0	1	64	0	2	28
1	0	126	1	1	206	1	2	54
2	0	217	2	1	285	2	2	94
3	0	223	3	1	337	3	2	91
4	0	240	4	1	316	4	2	99
5	0	188	5	1	278	5	2	75
6	0	161	6	1	205	6	2	59
7	0	100	7	1	143	7	2	33
8	0	69	8	1	81	8	2	22
9	0	33	9	1	44	9	2	7
10	0	19	10	1	18	10	2	3
11	0	5	11	1	7			
12	0	3	12	1	1			

Table 8

Overall quality of results obtained by the boson calculations in the fp shell

	1.0	0.5	0.2	0.1	0.05	EM
N	52	52	52	52	51	52
σ	1.687	1.621	1.411	1.115	0.497	0.962

$$\epsilon_{7/2} = 0,\ \epsilon_{3/2} = 2.07\text{MeV},\ \epsilon_{1/2} = 4.07\text{MeV}\ \epsilon_{5/2} = 6.0\text{MeV}. \tag{3}$$

Our aim has been to reproduce as accurately as possible the shell-model results for the same 52 low-lying states considered in the sd shell (see table 1), using some boson models containing only the most important (and low-lying) of the 60 possible bosons of the fp shell. Thus, in our main calculation, to be hereafter described as BM, we consider a boson space consisting of the s, d, g, i and d' bosons. The s, d, g and d' have been selected since they were found in the sd calculation, described in sect. 2, to be the four most important bosons for the description of low-lying states of the $A = 20$ nuclei. Therefore they are expected to play an important role in the fp shell as well. In addition the i ($J = 6$) boson, which is not present in the sd shell, has been included since it helps to account for states with relatively high spin. As shown in table 8, with these building blocks one can reproduce all of the 52 low lying states of table 1 with an rms deviation of 1.687.

There are two ways to improve this result. One way is to include more bosons in the model. To demonstrate the validity of this statement, we have attempted an IBM calculation, to be denoted by EM in the following, in a larger space than that used to obtain the BM results. Thus the space of the EM calculation contains, in addition to the s, d, g, i and d' bosons of BM, the s' boson, as well as the 4 $T = 0$ bosons lying lowest in energy, namely θ_1, θ_3, θ_5 and θ_7. As seen in table 8, the EM calculation reproduces the 52 low lying states of the $A = 44$ nuclei with a largely reduced rms deviation of 0.962.

Another way of improving the boson results, is the one described in sect. 3, i.e. by reducing the part of the fermion space taken into account. Considering the BM model space, we have repeated the calculation by taking into account for each (J, T) combination only the 50%, 20%, 10% and 5% of the low-lying fermion states of table 7. We observe in table 8 that the 50% reduction of the fermion space does not help much, but when we consider only the 10% or the 5% of the fermion space the results improve dramatically. A more detailed presentation of the results obtained by the BM calculation for the various truncations schemes of the fermion space is presented in table 9. For comparison we include in table 9 the shell-model and EM results.

Table 9
Energies of the low-lying states of ^{44}Ti (in MeV)
as a function of the number of fermion states included in the mapping

T	J_π	1.0	0.5	0.2	0.1	0.05	SM	EM
0	0_1	-11.21	-11.25	-11.40	-12.88	-12.88	-12.88	-12.59
	0_2	-5.57	-5.62	-6.27	-7.77	-8.38	-8.38	-7.00
	2_1	-10.06	-10.11	-10.27	-10.63	-11.37	-11.61	-10.79
	2_2	-7.99	-8.01	-8.18	-8.52	-8.89	-9.15	-8.26
	2_3	-4.19	-4.23	-4.55	-4.75	-5.94	-6.54	-4.59
	3_1	-5.60	-5.63	-5.77	-6.01	-6.50	-7.26	-6.05
	4_1	-8.74	-8.78	-8.93	-9.31	-10.36	-10.56	-9.44
	4_2	-6.23	-6.26	-6.48	-6.99	-7.71	-7.96	-6.87
	6_1	-7.78	-7.84	-8.07	-8.27	-9.08	-9.62	-8.47
	8_1	-6.36	-6.42	-7.23	-7.23	-7.23	-7.23	-6.75
1	1_1	-5.04	-5.12	-5.34	-5.59	-5.91	-6.45	-5.87
	2_1	-6.10	-6.13	-6.23	-6.44	-6.66	-7.26	-6.75
	2_2	-3.85	-3.88	-4.03	-4.24	-5.16	-5.45	-4.48
	3_1	-5.33	-5.38	-5.59	-5.82	-6.14	-6.51	-6.10
	3_2	-4.44	-4.59	-4.81	-5.03	-5.42	-5.97	-5.24
	4_1	-5.71	-5.74	-5.88	-5.95	-6.19	-6.88	-6.39
	4_2	-4.28	-4.31	-4.40	-4.60	-4.96	-5.31	-4.78
	5_1	-4.97	-5.07	-5.24	-5.42	-5.52	-6.26	-5.88
	6_1	-6.05	-6.09	-6.21	-6.36	-6.61	-6.94	-6.59
	7_1	-5.31	-5.55	-5.80	-5.88	-6.12	-6.27	-6.13
2	0_1	-4.53	-4.56	-4.64	-4.64	-4.64	-4.64	-4.55
	2_1	-3.02	-3.06	-3.13	-3.15	-3.15	-3.15	-3.03
	2_2	-1.41	-1.51	-1.56	-1.60	-1.61	-1.61	-1.46
	2_3	-1.22	-1.23	-1.32	-1.38	-1.38	-1.38	-1.25
	4_1	-2.19	-2.22	-2.27	-2.33	-2.33	-2.33	-2.19
	4_2	-1.93	-1.96	-1.99	-2.01	-2.01	-2.01	-1.93
	5_1	-1.07	-1.10	-1.14	-1.17	-1.19	-1.19	-1.07
	6_1	-1.67	-1.70	-1.72	-1.78	-1.78	-1.78	-1.67

5. Conclusions

In this paper we have attempted an application of the democratic mapping in the case of the sd and fp shells. This application demonstrated the applicability of the method in realistic cases of several non-degenerate orbitals. The main conclusions of this work are summarized here:

i) We have demonstrated in a completely impartial way that the s and d bosons are the most essential building blocks of the low lying states in sd shell nuclei. Thus, one of the main *assumptions* of IBM is *proven* to be correct. The d' and g bosons have been found to be the next most important ones, according to expectations [17–20].

ii) Very accurate results have been obtained in the sd shell by considering the 12 lowest lying bosons out of the 28 possible ones. However, to obtain an equally good agreement in the fp shell one needs to enlarge considerably the dimension of the boson space. This result is a consequence of the democratic mapping method which treats all fermion states which are mapped onto boson states on equal footing.

iii) One way to obtain boson results in good agreement with the shell model calculation using relatively few bosons (4 bosons in the sd shell, 5 bosons in the fp shell) is to map only the fermion subspace which contains the states of interest. Although the results of this approach resemble those obtained by the OAI mapping method, still one is not required to make any assumptions about the shell-model states to be mapped. Thus the only requirement considered in sections 3 and 4 was that the fermion states to be mapped are the low-lying ones. Equally well one could have applied the mapping procedure to reproduce the energies of shell-model states selected in some other fashion.

Concerning plans for future work along these lines, it should be noticed that the importance of higher order terms has been recently realized in both the algebraic [33–35] and the shell model [36,37] framework. An effort is therefore under way to include such higher order terms in the framework of the democratic approach.

References

[1] A. Arima and F. Iachello, Phys. Rev. Lett. **35** (1975) 1069

[2] A. Arima and F. Iachello, Ann. Phys. **99** (1976) 253; **111** (1978) 201; **123** (1979) 468

[3] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987)

[4] D. Bonatsos, *Interacting Boson Models of Nuclear Structure* (Clarendon, Oxford, 1988)

[5] A. Klein and E. R. Marshalek, Rev. Mod. Phys. **63** (1991) 375

[6] T. Otsuka, A. Arima and F. Iachello, Nucl. Phys. **A309** (1978) 1

[7] A. de Shalit and I. Talmi. *Nuclear Shell Theory* (Academic Press. New York. 1963)

[8] T. Marumori. M. Yamamura and A. Tokunaga, Progr. Theor. Phys. **31** (1964) 1009
T. Marumori. M. Yamamura, A. Tokunaga and T. Takada, Progr. Theor. Phys. **32** (1964) 726

[9] S. T. Belyaev and V. G. Zelevinsky. Nucl. Phys. **39** (1962) 582

[10] E. R. Marshalek. Nucl. Phys. **A224** (1974) 221, 245

[11] D. Bonatsos. A. Klein and C. T. Li. Nucl. Phys. **A425** (1984) 521
D. Bonatsos and A. Klein, Nucl. Phys. **A469** (1987) 253

[12] D. P. Menezes. D. Bonatsos and A. Klein, Nucl. Phys. **A474** (1987) 381
D. Bonatsos. D. P. Menezes and A. Klein. J. Phys. G **14** (1988) L45
D. P. Menezes. D. Phil. thesis (Oxford, 1989)

[13] D. P. Menezes. N. Yoshinaga and D. Bonatsos, Phys. Lett. **221B** (1989) 103
D. P. Menezes and N. Yoshinaga. Nucl. Phys. **A517** (1990) 264
D. P. Menezes, Rev. Bras. Fis. **20** (1990) 336

[14] L.D. Skouras. P. Van Isacker and M.A. Nagarajan, Nucl. Phys. **A516** (1990) 255

[15] L.D. Skouras and S. Kossionides. Comp. Phys. Comm. **39** (1986) 197

[16] S. Kossionides and L.D. Skouras. Comp. Phys. Comm. **39** (1986) 213

[17] H. C. Wu, Phys. Lett. **110B** (1982) 1

[18] Y. Akiyama. Nuc. Phys. **A433** (1985) 369

[19] N. Yoshinaga. Y. Akiyama and A. Arima, Phys. Rev. Lett. **56** (1986) 1116

[20] P. Van Isacker. K. Heyde. M. Waroquier and G. Wenes, Nucl. Phys. **A380** (1982) 383

[21] J. P. Elliott and A. P. White, Phys. Lett. **97B** (1980) 169

[22] J. P. Elliott and J. A. Evans, Phys. Lett. **101B** (1981) 216

[23] K. Heyde. Int. J. Mod. Phys. **A4** (1989) 2063

[24] P. Halse. J. P. Elliott and J. A. Evans, Nucl. Phys. **A417** (1984) 301

[25] P. Halse. Nucl. Phys. **A445** (1985) 93

[26] N. Yoshinaga. Nucl. Phys. **A493** (1980) 323

[27] J. Engel and F. Iachello. Phys. Rev. Lett. **54** (1985) 1126; Nucl. Phys. **A472** (1987) 61

[28] N. Yoshinaga and D. M. Brink, Nucl. Phys. **A515** (1990) 1

[29] F. Ajzenberg-Selove. Nucl. Phys. **A460** (1986) 1

[30] B.M. Preedom and B.H. Wildenthal, Phys. Rev. **C6** (1972) 1633

[31] M. Abdelaziz. J.P. Elliott. M.J. Thompson and J.A. Evans, Nucl. Phys. **A503** (1989) 452

[32] T.T.S. Kuo and G.E. Brown, Nucl. Phys. **A114** (1968) 241

[33] G. Vanden Berghe, H. E. De Meyer and P. Van Isacker, Phys. Rev. **C32** (1985) 1049

[34] D. Bonatsos. Phys. Lett. **200B** (1988) 1

[35] J. Vanthournout, Phys. Rev. **C41** (1990) 2380

[36] L.D. Skouras and J.C. Varvitsiotis. Nucl. Phys. **A513** (1990) 264

[37] A. G. M. van Hees. J. G. L. Booten and P. W. M. Glaudemans, Nucl. Phys. **A507** (1990) 55c