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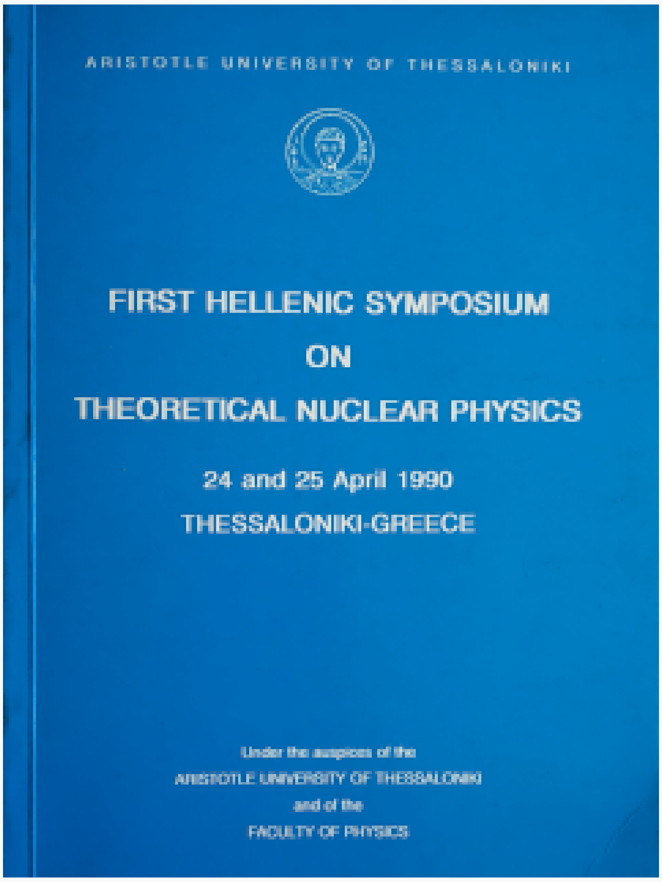

A completely integrable case in the complex Lorenz equations

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A completely integrable case in the complex Lorenz equations *

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Abstract

By application of the Lie theory of extended groups and for the parameter values $\sigma = 1/2$, $b = 1$, $r_1 = e^2/2$, $r_2 = e/2$, e arbitrary we prove that the system of the complex Lorenz equations is algebraically completely integrable. The respective general exact solution is expressed by means of Jacobian elliptic functions.

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The non-linear system of differential equations, the complex Lorenz equations (CLE),

$$\dot{x} = \sigma(y-x) \tag{1a}$$

$$\dot{y} = rx - xz - az \tag{1b}$$

$$\dot{z} = -bz + (x^*y + xy^*)/2 \tag{1c}$$

were first proposed and subsequently numerically investigated by Fowler, Gibbon and McGuinness [1-3]. The dot in eqs.(1a)-(1c) denotes differentiation with respect to the time t and the parameters b, σ, r, a are defined by

$$b>0,\ \sigma>0,\ r = r_1 + ir_2,\ r_1>0,\ r_2>0,\ a = 1 - ie,\ e>0 \tag{2}$$

The dynamical system (1a)-(1c) passes the Painlevé test for

$$\sigma = 1/2,\ b = 1,\ r_1 = e^2/2,\ r_2 = e/2,\ e \text{ arbitrary} \tag{3a}$$

$$\sigma = 1,\ b = 2,\ r_1 = e^2/4 + 1/9,\ r_2 = 0,\ e \text{ arbitrary} \tag{3b}$$

$$\sigma = 1/3,\ b = 0,\ r_1 \text{ arbitrary},\ r_2 = e,\ e \text{ arbitrary} \tag{3c}$$

as shown by Roekaerts [4]. Although constants of the motion of eqs(1a)-(1c) of the form

$$F(x_1,x_2,x_3,x_4,z,t) = \exp(c_0 t)P(x_1,x_2,x_3,x_4,z) \tag{4a}$$

$$x = x_1 + ix_2\ ,\quad y = x_3 + ix_4\ , \tag{4b}$$

where c_0 is a constant depending on σ, b, r_1, r_2, e and P is a polynomial of at most fourth order in its arguments, have been recently constructed [4], the important possibility of the system (1a)-(1c) being algebraically completely integrable for some parameter values has up till now not been addressed to. We note here that it is very rare for a non-linear dynamical system depending on parameters and deriving from a physical problem to be completely integrable even for specific parameter values [5-6]. On the other hand a thorough numerical investigation of the CLE, which serve as a model of dispersively unstable, weakly non - linear, weakly damped physical systems, is practically a very hard task due to the number of independent parameteres [1-3]. Consequently the search for completely integrable cases valid for specific values of the parameters, on which a given dynamical system depends, seems to be worth pursuing since the corresponding general exact solutions, apart from the importance they may have in their own right, can also be used to test the feasibility of numerical algorithms.

In the present work we present the general exact solution to eqs.(1a)-(1c) for the Painlevé case (3a). We recall at this point that passing the Painlevé test is only a necessary and by no means a sufficient condition for complete integrability through Abelian functions [5]. We need only mention that there are dynamical systems which do not pass the Painlevé test and yet are completely integrable [5]. In the following, however, by constructing the general solution of eqs.(1a)-(1c) by means of Jacobian elliptic functions and for the parameter values (3a), we shall demonstrate that passing the Painleve test for the case (3a) is a necessary and sufficient condition for algebraic complete integrability of the CLE.

The system of eqs.(1a)-(1c) can be reduced to the equivalent system

$$\ddot{x} + (\sigma+a)\dot{x} + \sigma(a-r)x = -\sigma xz \tag{5a}$$

$$\dot{z} + zb = |x|^2 + \left[d(|x|^2)/dt\right](1/2\sigma) \tag{5b}$$

$$y = \dot{x}/\sigma + x. \tag{5c}$$

From eq.(5b) we get

$$z(t) = C\exp(-bt) + |x|^2/2\sigma + \exp(-bt)(1 - b/2\sigma)\int |x|^2 \exp(bt)dt. \tag{6}$$

C being a constant of integration.

Let now

$$b = 2\sigma \tag{7}$$

Equations (5a) and (6) - (7) yield

$$\ddot{x} + (\sigma + a)\dot{x} + \sigma(a - r)x = -x|x|^2/2 - \exp(-2\sigma t)C\sigma x. \tag{8}$$

On introducing the functions $u = u(\xi)$, $\xi = \xi(t)$, $v = v(t)$ through

$$x(t) = u(\xi)v(t) \tag{9}$$

eq.(8) becomes after some algebra and, essentially, by solving a differential equation of the type of Bessel for $v(t)$,

$$u'' = -\exp[-(\sigma + 1)t]u|u|^2 Z_p^4(\zeta)|Z_p(\zeta)|^2/2, \tag{10a}$$

the dash denoting differentiation with respect to ξ, and

$$\zeta = (C/\sigma)^{\frac{1}{2}}\exp(-\sigma t),\ C > 0 \tag{10b}$$

$$v(t) = \exp[-(\sigma + a)t/2]Z_p(\zeta) \tag{10c}$$

$$\dot{\xi}(t) = \exp[-(\sigma + a)t]/v^2, \tag{10d}$$

where $Z_p(\zeta)$ is a cylinder function of order p and

$$p = [(\sigma - a)^2 + 4\sigma r]^{\frac{1}{2}}/2\sigma. \tag{10e}$$

We consider now $p = 1/2$. Then eq.(10e) yields the following relations between the parameters σ, e, r_1, r_2 :

$$1 + 2\sigma(2r_1 - 1) - e^2 = 0 \tag{11a}$$

$$e(\sigma - 1) + 2\sigma r_2 = 0. \tag{11b}$$

Thus, since $Z_{1/2}(\zeta) = J_{1/2}(\zeta)$, where $J_{1/2}(\zeta)$ is the Bessel function of the first type, eq.(10a) becomes

$$u'' = -\lambda u|u|^2\exp[(2\sigma - 1)t]\sin^6\zeta,\ \lambda = (4/n^3)(\sigma/C)^{3/2} \tag{12a}$$

$$\xi(t) = (\cot\zeta)n/2\sigma,\ \zeta = \exp(-\sigma t)(C/\sigma)^{\frac{1}{2}}, \tag{12b}$$

eq.(12b) following from eqs.(10c)-(10d) with $p = 1/2$.

On setting in eqs.(12a)-(12b) $\sigma=1/2$ and by introducing a function $F(\rho)$ through

$$u(\xi) = F(\rho),\ \rho = \xi/n = \cot\zeta,\ \zeta=\exp(-t/2)(2C)^{\frac{1}{2}} \tag{13a}$$

eq.(12a) is written as

$$F''(\rho) = -\epsilon F(\rho)|F(\rho)|^2/(\rho^2 + 1)^3, \ \epsilon = \bar{A}n^2, \ \bar{A} = (4/n^3)(2C)^{-3/2} \tag{13b}$$

provided

$$\sigma = 1/2, \ b = 1, \ r_1 = e^2/2, \ r_2 = e/2, \ e \text{ arbitrary} \tag{14}$$

due to conditions (11a)-(11b).

We distinguish two cases :

Case (A) : $u(\xi)$ complex. Let

$$F(\rho) = F_1(\rho) + iF_2(\rho) \tag{15}$$

with real $F_1 = F_1(\rho)$, $F_2 = F_2(\rho)$. By virtue of eq.(15) eq.(13b) gives

$$F_1'' = -\epsilon F_1(F_1^{\,2} + F_2^{\,2})/(\rho^2 + 1)^3 \tag{16a}$$

$$F_2'' = -\epsilon F_2(F_1^{\,2} + F_2^{\,2})/(\rho^2 + 1)^3. \tag{16b}$$

Excluding for the time being the trivial solutions (α) $F_1 = 0$ and (β) $F_1 = F_2$ we may find the general solution of the system (16a) (16b) by setting

$$F_1 = R(\rho)\cos[\varphi(\rho)], \quad F_2 = R(\rho)\sin[\varphi(\rho)]. \tag{17}$$

Upon insertion of eq.(17) into eqs.(16a)-(16b) we find that eqs.(16a)-(16b) are satisfied if

$$R'' + \epsilon R^3/(\rho^2 + 1)^3 - C_1^{\,2}/R^3 = 0, \ R = R(\rho) \tag{18a}$$

$$\varphi' = C_1/R^2, \ \varphi = \varphi(\rho), \ C_1 \text{ constant.} \tag{18b}$$

Thus we have reduced our problem to the solution of the non-linear differential equation (18a). We shall treat eq.(18a) in the framework of the Lie theory of extended groups [6-8]. In fact, eq.(18a) has the symmetry

$$G = \xi(\rho,R)\partial/\partial\rho + \eta(\rho,R)\partial/\partial R, \ R = R(\rho) \tag{19}$$

provided [6-8]

$$G^{(2)} N(R'', R, p) = 0 \tag{20a}$$

where

$$N(R'', R, p) = R'' + \varepsilon R^3/(p^2 + 1)^3 - C_1{}^2/R^3 = 0 \tag{20b}$$

and $G^{(2)}$ is the second extention of G given by

$$G^{(2)} = G + (\eta' - \xi' R')\partial/\partial R' + (\eta'' - \xi'' R' - 2\xi' R'')\partial/\partial R'' \tag{20c}$$

with $\eta' = d\eta/dp$, $\xi' = d\xi/dp$. Equations (19)-(20c) yield

$$\xi(p,R)\partial g/\partial p + \eta(p,R)\partial g/\partial R + (\eta'' - \xi'' R' - 2\xi' R'') = 0 \tag{21a}$$

where

$$g = g(R,p) = \varepsilon R^3/(p^2 + 1)^3 - C_1{}^2/R^3. \tag{21b}$$

We obtain from eqs.(21a)-(21b) after separating coefficients of $(R')^3$, $(R')^2$, R', R, as both ξ and η are functions of p and R only,

$$\partial^2\xi/\partial R^2 = 0 \tag{22a}$$

$$\partial^2\eta/\partial R^2 - 2\partial^2\xi/\partial R\partial p = 0 \tag{22b}$$

$$2\partial^2\eta/\partial R\partial p - \partial^2\xi/\partial p^2 + 3g\,\partial\xi/\partial R = 0 \tag{22c}$$

$$\partial^2\eta/\partial p^2 - g\,\partial\eta/\partial R + 2g\,\partial\xi/\partial p + \xi\,\partial g/\partial p + \eta\,\partial g/\partial R = 0. \tag{22d}$$

Solution of the system of partial differential equations (22a)-(22d) can be carried out and we deduce

$$\xi(p,R) = p^2 + 1, \quad \eta(p,R) = pR \tag{23}$$

and eq.(19) shows that

$$G = (p^2 + 1)\partial/\partial p + (pR)\partial/\partial R. \tag{24}$$

The generator G in eq.(24) can be transformed to $\partial/\partial P$ by means of the transformation [6-8]

$$P = \tan^{-1}p, \quad f = R(p^2 + 1)^{-1/2} \tag{25}$$

in which case eq.(18a) is written owing to eq.(25) as

$$f'' + \varepsilon f^3 + f - C_1{}^2/f^3 = 0,\ f = f(P). \tag{26}$$

Equation (26) has the first integral

$$(f')^2 + \varepsilon f^4/2 + f^2 + C_1{}^2/f^2 = C_2,\ C_2>0 \tag{27}$$

and from eq.(27) we conclude that

$$\int_{f_0}^{f} \left[C_2 - \varepsilon f_1{}^4/2 - f_1{}^2 - C_1{}^2/f_1{}^2\right]^{-1/2} df_1 = P - P_0\ ,\ f(P_0) = f_0 \tag{28}$$

Depending on the values of the constants C_1, C_2, ε three different cases arise, all of which lead essentially to the result, that the integral in eq.(28) is expressible by means of elliptic functions. We shall give below the final result which derives from one of the three cases above, namely when the polynomial $\varphi(w) = C_2 w - \varepsilon w^3/2 - w^2 - C_1{}^2 > 0$, $w = f_1{}^2$, has one real root z_1. Thus we obtain from eqs.(13a), (15), (17), (25), (28), after inverting the elliptic integral,

$$u(\xi) = [z_1 - \tau\, sn^2\Lambda/(1 + cn\Lambda)^2](1 + \xi^2/n^2)^{1/2} \exp[i\varphi(\xi/n)], \tag{29}$$

where z_1 and $z_2 = \alpha + i\beta$, $z_3 = \alpha - i\beta$ are the real and complex roots of $\varphi(w)$ respectively,

$$\Lambda = \Lambda(\xi) = F(\gamma_2,\delta) + [P_0 - \tan^{-1}(\xi/n)](\varepsilon\tau/2)^{1/2} \tag{30a}$$

$$\tau = [(\alpha - z_1)^2 + \beta^2]^{1/2},\ \xi/n = \cot[\exp(-t/2)(2C)^{1/2}] \tag{30b}$$

$$\gamma_2 = 2\tan^{-1}[((z_1 - f_0)/\tau)^{1/2}],\ \delta = \left[(\tau - \alpha + z_1)/2\tau\right]^{1/2},\ f_0 < z_1. \tag{30c}$$

In eq.(29) $sn\Lambda$, $cn\Lambda$ are the Jacobian elliptic functions and in eq.(30a) $F(\gamma_2,\delta)$ is the elliptic integral of the first kind. As now from eqs(10c), (12b) with $p=1/2$, $\sigma = 1/2$, we get

$$v(t) = \exp[t(fe - 1)/2]\sin[(2C)^{1/2}\exp(-t/2)](2/n^2C)^{1/4} \tag{31}$$

the general exact solution to eqs.(1a)-(1c) can be easily written down from eqs.(9), (5c), (6) for the values of the parameters in eq.(14). The function $\varphi(\xi/n)$ is of no further importance since it is only a phase angle.

Case (B) : $u(\xi)$ real

On setting in eq.(15) $F_2(\rho) = 0$ it only remains from eqs.(16a)-(16b) to solve

$$F_1'' = -\varepsilon F_1{}^3/(\rho^2 + 1)^3. \tag{32}$$

Furthermore, the trivial solutions mentioned after eqs.(16a)-(16b) lead to equations of the form (32). Equation (32) can be solved in the context of extended groups [6-8] and we obtain its general solution by means of elliptic functions.

In summary we have constructed the general exact solution to the CLE for the parameter values (3a) by means of Jacobian elliptic functions. Evidently the above solution possesses the Painlevé property and approaches for $t \to \infty$ the stable equilibrium point $x = y = z = 0$. Therefore we have proved that passing the Painlevé test for the case (3a) is indeed a necessary and sufficient condition for the CLE to be algebraically completely integrable. Finally we note that the case (3b) can also be treated by the method of the present work by considering $p=1/3$ in eq.(10e) and using Airy functions.

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