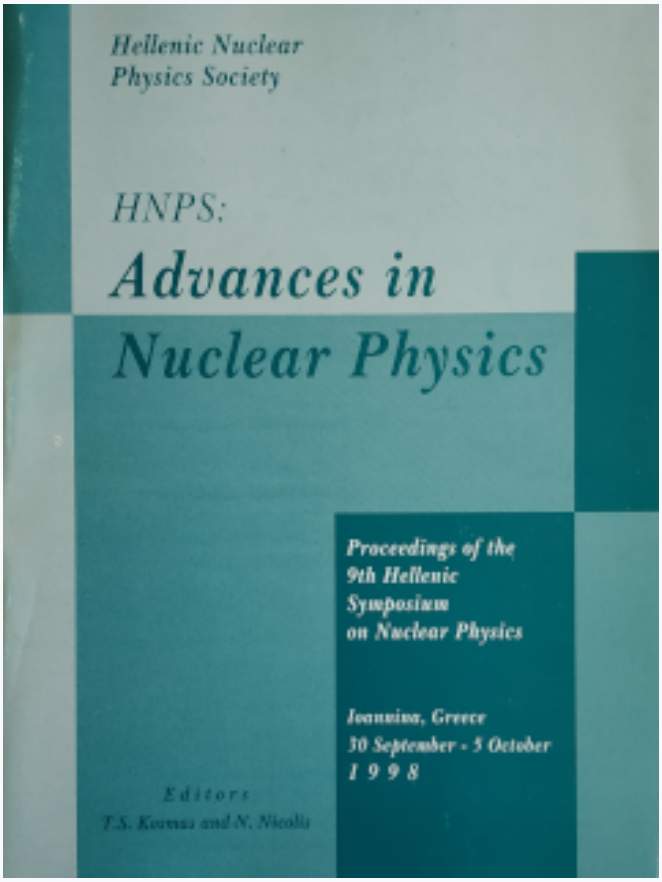
HNPS Advances in Nuclear Physics

Vol 9 (1998)

HNPS1998



Fundamental nuclear structure symmetries in double beta decay processes

O. Civitarese

doi: 10.12681/hnps.2792

Fundamental nuclear structure symmetries in double beta decay processes

Osvaldo Civitarese

Department of Physics, University of La Plata, c.c. 67 (1900), La Plata, Argentina.

Abstract

The nuclear structure physics of double beta decay transitions is reviewed starting from the consideration of fundamental symmetries of the nuclear many body problem. The problems found in the use of the Quasiparticle Random Phase Approximation (QRPA) and related approximations, in dealing with the calculation of nuclear double beta decay observables, are understood in terms of the mixing between isospin collective and intrinsic variables.

1 Introduction

Years ago (1988) John Vergados did in fact point out to a number of us that the understanding of the nuclear structure physics which governs nuclear double beta decay transitions was still a challenging matter and that the currently accepted explanations were in his opinion not very satisfactory. He was indeed right and after ten years the subject remains a hot one, attracting the attention of a very distinguished community of theoretical and experimental groups. I would like to present in this lecture some answers to John's questions as my very modest tribute to his life-long efforts in the fascinating field of the interface between particle and nuclear physics.

2 Spontaneous and Dynamical Breaking of Mean Field Symmetries in the pn-QRPA and the description of Double Beta Decay transitions

The motivation for this part of the lecture are the breakdown of the isospin symmetry, at the level of the quasi-particle mean field approximation, and its

partial restoration by effective interactions, at the QRPA level of approximation. The method upon which the discussion is based has been used to define effective symmetry breaking two-body interactions and it has been applied previously to particle-number and rotational symmetry violations. We shall also talk on the connection between the present approach and the proton-neutron QRPA method with renormalized two-particle interactions is discussed.

The formalism of the Quasiparticle Random Phase Approximation has been used rather intensively and with great success during four decades since the fundamental paper of Michel Baranger appeared in the literature [1]. The various applications of the formalism, to describe nuclear vibrational modes, are matter of textbooks [2–4]. The use of the QRPA method to describe charge-dependent excitations (pn-QRPA) was reported by Halbleib and Sorensen [5]. Symmetry properties of the approximation can be found in articles written by Lane and Martorell [6] and by Marshalek and Weneser [7]. In the original version of the QRPA [1] the nuclear many body Hamiltonian consists of short-range pairing interactions and residual two body interactions and it is written in the quasiparticle basis (BCS approach) and diagonalized in the quasiparticle-pair basis. The structure of the ground state correlations generated at the QRPA level of approximation was studied in [4]. More recently the use of renormalized two-particle channels of the residual interactions at the level of the pn-QRPA matrix elements was suggested by Vogel and Zirnbauer [8]. The application of these concepts to realistic calculations of double beta decay observables can be found in [9]. After several years of theoretical efforts centered on the use of the pn-QRPA method to calculate single- and double-beta-decay observables [10,11] some questions associated with the consistency of the approach have been revised, partly due to some considerations about the collapse of the QRPA (pn-QRPA) approximation [12,13]. Studies of this question, performed in the framework of group theoretical models, have been presented in [14]. The analysis of the pn-QRPA collapse in terms of a phase transition in a parametric model space was presented in [15,16]. Recent results based on the separation of intrinsic and collective variables [17] confirm the notion that the standard formulation of the pn-QRPA method should be extended. Among the basic theoretical assumptions which should indeed be revisited, in dealing with the explanation of the pn-QRPA collapse are:

- the separate treatment of proton and neutron isovector pairing correlations, which are usually represented by unrelated BCS mean fields belonging to the initial and final double-even mass nuclei,
- the onset of isoscalar pairing correlations, affecting both the double-even and double-odd mass nuclei,
- the resulting violation of the isospin symmetry once the proton–proton and neutron–neutron BCS procedure is applied to describe, approximately, the pairing correlations.

All these effects would certainly become manifest at mean field (quasiparticle) level [18,19]. In addition to these effects, which are generally referred to as spontaneous symmetry violations, one should add the fact that empirical single particle basis are used as input for the pn-QRPA calculations, thus contributing to undesirable symmetry violations. As it has been pointed out long ago, the QRPA by itself may not be able to cure for the resulting mean field symmetry violations [6]. The relationship between the collapse of the pn-QRPA and the onset of isoscalar pairing correlations was discussed in [14], in the framework of the SO(8) global symmetry of the Hamiltonian. The spontaneous breaking of the isospin symmetry, induced by the separate BCS treatment of proton and neutron pairing correlations is rather obvious. In this respect, the inclusion of symmetry violating interactions may be crucial in treating isospin dependent effects [17]. In this part of the lecture we discuss on symmetry violation effects in the pn-QRPA by using a method due to Pyatov [20]. The main step of Pyatov's construction is the definition of an effective Hamiltonian which incorporates Dirac's constrains [21] to the original symmetry-breaking Hamiltonian. This method has been used previously in dealing with the violation of particle-number [22], rotational [23] and generalized Galilean invariances [24] and velocity dependent effects [25]. It is our aim to apply Pyatov's method to isospin dependent Hamiltonians, written in the quasiparticle basis, in order to explore the link between the collapse of the pn-QRPA and the breakdown of the isospin symmetry. Particularly, we would like to determine the dependence of the renormalization applied to two particle interactions, in the pn-QRPA upon symmetry restoring effects.

2.1 The BCS mean field and the symmetry restoring interactions

The separable monopole pairing interaction can be approximately diagonalized in the BCS quasiparticle representation. The proton and neutron single quasiparticle Hamiltonian can therefore be written as

$$H_{11} = \sum_j (E_{pj} N_{pj} + E_{nj} N_{nj}) \ , \tag{1}$$

where the one-quasiparticle term $N_{q,j}$ is written in standard notation [3]

$$N_{q,j} = \sum_m \alpha^\dagger_{q,jm} \alpha_{q,jm} \ . \tag{2}$$

The subindex q denotes proton (p) or neutrons (n) states and (j, m) are single particle angular momentum variables.

The fact that the isospin symmetry is violated by the BCS quasiparticle mean field can be easily demonstrated by expressing the tensorial components (total angular momentum $\lambda = 0$, isospin $\tau = 1$) [26,2] of the isospin one body operator in the quasiparticle basis

$$\tau^{(-)} = \sum_j (t_j A_j^\dagger + \bar{t}_j A_j) \,,$$
$$\tau^{(+)} = \sum_j (t_j A_j + \bar{t}_j A_j^\dagger) \,, \tag{3}$$

and commuting it with the unperturbed BCS Hamiltonian (1). Note that only the two-quasiparticle terms of the isospin operators $\tau^\pm$

$$A_j^\dagger = \frac{1}{\sqrt{(2j+1)}} \sum_{m>0} \alpha_{p,jm}^\dagger \alpha_{n,\overline{jm}}^\dagger \,,$$
$$A_j = (A_j^\dagger)^\dagger \,, \tag{4}$$

will contribute to the expectation value of the commutators.

In (1) E_{pj} and E_{nj} are quasiparticle energies and the reduced matrix elements in (3) are defined as

$$t_j = \sqrt{(2j+1)} u_{pj} v_{nj}, \quad \bar{t}_j = \sqrt{(2j+1)} u_{nj} v_{pj} \,. \tag{5}$$

The results corresponding to the commutators

$$[H_{11}, \tau^{(-)}] = \Theta^{(-)} \,,$$
$$[H_{11}, \tau^{(+)}] = \Theta^{(+)} = -(\Theta^{(-)})^\dagger \,, \tag{6}$$

lead to the definition of the operator

$$\Theta = \frac{1}{2}(\Theta^{(-)} + \Theta^{(+)}) = \frac{1}{2} \sum_j E_j (t_j + \bar{t}_j)(A_j^\dagger - A_j) \,, \tag{7}$$

where $E_j = E_{pj} + E_{nj}$. Following Pyatov's Method [20], the effective Hamiltonian which exhibits the symmetry can be constructed from the above commutator adding to H_{11} the induced interaction

$$H_{\text{res}} = -\gamma \Theta^\dagger \Theta \,. \tag{8}$$

The value of γ is determined by requesting that the zero energy mode [7] is decoupled from the physical spectrum of $H_{\text{eff}} = H_{11} + H_{\text{res}}$, as it will be shown below.

The QRPA treatment of H_{eff}, is performed by introducing the one-phonon operator $\Gamma^{\dagger}_{\nu}$

$$\Gamma^{\dagger}_{\nu} = \sum_{k} (X_{\nu k} A^{\dagger}_{k} - Y_{\nu k} A_{k}) \,, \tag{9}$$

and in this basis the QRPA equation of motion is written

$$\left[H_{\text{eff}}, \Gamma^{\dagger}_{\nu} \right] = \omega_{\nu} \Gamma^{\dagger}_{\nu} \,. \tag{10}$$

The solution of this equation of motion, in the quasi-boson approximation $[A_j, A^{\dagger}_{k}] = \delta_{jk}$, can be cast in the form

$$(1 - s_{11}) = 0 \,. \tag{11}$$

The quantity s_{11} is defined by

$$s_{11} = \frac{\gamma}{2} \sum_{j} E_j^2 (t_j + \bar{t}_j)^2 \left(\frac{1}{E_j - \omega_{\nu}} + \frac{1}{E_j + \omega_{\nu}} \right) \,. \tag{12}$$

After solving the QRPA system of equations, in order to determine the eigenfrequencies ω_{ν} and the amplitudes $X_{\nu k}$ and $Y_{\nu k}$ of (9), H_{eff} is transformed to the phonon basis

$$H_{\text{eff}} = \text{const.} + \sum_{\nu} \omega_{\nu} \Gamma^{\dagger}_{\nu} \Gamma_{\nu} \,. \tag{13}$$

The decoupling, at the QRPA level of approximation, of the zero energy mode can be performed by introducing the transformation due to Marshalek and Weneser [7]:

$$\begin{aligned} \hat{P}_{\nu} &= (\omega_{\nu}/2)^{1/2} (\Gamma^{\dagger}_{\nu} + \Gamma_{\nu}) \,, \\ \hat{L}_{\nu} &= -i (2\omega_{\nu})^{-1/2} (\Gamma^{\dagger}_{\nu} - \Gamma_{\nu}) \,. \end{aligned} \tag{14}$$

The expression of H_{eff} in terms of the operators $\hat{P}_{\nu}$ and $\hat{L}_{\nu}$ is given by

$$H_{\text{eff}} = \frac{1}{2} \sum_{\nu} (\hat{P}^2_{\nu} + \omega^2_{\nu} \hat{L}^2_{\nu}) \,. \tag{15}$$

This diagonal form of H_{eff} is obtained by transforming the pair operators $A^{\dagger}_{j}$ and A_j to the phonon basis ($\Gamma^{\dagger}_{\nu}$, Γ_{ν}) and then to the basis ($\hat{P}_{\nu}$, $\hat{L}_{\nu}$). The

explicit expression of H_{eff} is

$$H_{\text{eff}} = \sum_{\nu\mu}(d_{\nu\mu}\hat{P}_\nu\hat{P}_\mu + c_{\nu\mu}\omega_\nu\omega_\mu\hat{L}_\nu\hat{L}_\mu) \ , \tag{16}$$

with

$$\begin{aligned} d_{\nu\mu} &= \sum_{kl}(E_k\delta_{kl} + (V_{kl} + W_{kl}))\lambda_{\nu k}\lambda_{\mu l} \ , \\ c_{\nu\mu} &= \sum_{kl}(E_k\delta_{kl} + (V_{kl} - W_{kl}))\mu_{\nu k}\mu_{\mu l} \ . \end{aligned} \tag{17}$$

In these equations we have defined

$$\begin{aligned} \lambda_{\nu k} &= \frac{(X_{\nu k} + Y_{\nu k})}{\sqrt{2\omega_\nu}} \ , \\ \mu_{\nu k} &= \frac{(X_{\nu k} - Y_{\nu k})}{\sqrt{2\omega_\nu}} , \\ V_{kl} &= -W_{kl} = -\frac{\gamma}{4}E_kE_l(t_k + \bar{t}_k)(t_l + \bar{t}_l) \ . \end{aligned} \tag{18}$$

The diagonalization of H_{eff} implies $d_{\mu\nu} = c_{\mu\nu} = \frac{1}{2}\delta_{\mu\nu}$ and the decoupling of the zero energy mode requires that

$$\gamma \equiv \gamma_0 = 1/(\sum E_k(t_k + \bar{t}_k)^2) \ . \tag{19}$$

With this value of γ, the QRPA secular equation, (11), takes the form

$$\omega_n^2 F(\omega_\nu) = 0 \ ,$$

which is obviously satisfied for $\omega_\nu^2 = 0$ (zero-energy eigenmode) and $F(\omega_\nu) = 0$ ($\omega_\nu \neq 0$). The explicit form of $F(\omega_\nu)$ is the following :

$$F(\omega_\nu) = \sum_k \frac{E_k(t_k + \bar{t}_k)^2}{(E_k^2 - \omega_\nu^2)} \ . \tag{20}$$

2.2 Separable particle–hole and particle-particle pn-interactions

Results corresponding to realistic proton-neutron interactions in open shell systems have been compared rather successfully with results obtained by using schematic interactions of the form [27,28]

$$H = H_{11} + H_{\text{int}} \ , \tag{21}$$

with

$$H_{\rm int} = \chi\tau^{(-)}\tau^{(+)} - \kappa P^{(-)}P^{(+)} \ . \tag{22}$$

The proton neutron pair operators $P^{(\pm)}$ are written in the quasiparticle basis as

$$\begin{aligned} P^{(-)} &= \sum_j (p_j A_j^\dagger - \overline{p}_j A_j) \ , \\ P^{(+)} &= \sum_j (p_j A_j - \overline{p}_j A_j^\dagger) \ , \end{aligned} \tag{23}$$

with

$$\begin{aligned} p_j &= \sqrt{(2j+1)} u_{pj} u_{nj} \ , \\ \overline{p}_j &= \sqrt{(2j+1)} v_{nj} v_{pj} \ , \end{aligned} \tag{24}$$

and they represent particle-particle (hole-hole) terms of the one particle operator τ. The term of the Hamiltonian (22) which contains the pair operators $P^{\pm}$ will be referred to as the particle-particle interaction. Solutions corresponding to the Hamiltonian of (21) have been obtained both exactly and approximately [15]. Since details of these calculations have been presented previously [15,28] further discussions about this Hamiltonian will be avoided.

Numerical results corresponding to the Hamiltonians of Secs. 2.1 and 2.2 are presented in [39]. In Sec. 2.1 we have shown that the BCS mean field does not preserve the isospin symmetry and we have used Pyatov's Method [20] to partially restore it at the two-quasiparticle level of approximation. In Sec. 2.2 we have introduced an effective Hamiltonian which obviously breaks the isospin symmetry. The pn-QRPA treatment of this Hamiltonian leads to collapse of the approximation for some values of κ. In this context the question to ask is, of course, to which extend these features survive if the BCS mean field is readjusted in such a way that the spontaneously broken isospin symmetry is partially or totally restored in a dynamical way (i.e. by adding terms as the ones of $H_{\rm int}(\chi,\kappa)$). The first obvious answer to such a question would refer to limitations in the values of the renormalized particle-particle constant κ resulting from the inclusion of terms depending on γ in the Hamiltonian.

2.3 Symmetry Restoring Effects at the BCS mean field level

The most general form of the Hamiltonian, at lowest order in the quasi-boson expansion (i.e. by keeping terms with $A^\dagger A$, $A^\dagger A^\dagger$ and AA), contains terms

which are proportional to γ, χ and κ. Effects associated to the spontaneous symmetry breaking of the isospin symmetry by the BCS quasiparticle mean field are explored by studying the dependence of the QRPA spectrum upon γ. Note that as $\gamma \to \gamma_0 = 1/(\sum_k E_k(t_k + \bar{t}_k)^2)$ the lowest eigenvalue goes to zero. This result is the direct consequence of the use of Pyatov's prescription [20]. The strength $\gamma = \gamma_0$ represents the value of the induced interaction (8) which restores the isospin symmetry, broken by the BCS approach, at the quasi-boson level. Naturally the breakdown of the symmetry is due to the adoption of separate quasi-neutron and quasi-proton mean fields and it is obviously non-physical.

Since γ_0 represents the value of the induced coupling for which the symmetry is restored, it can be argued that the inclusion of residual interactions (χ, κ), for partial restoration ($\gamma < \gamma_0$), would break the symmetry dynamically. An example of this mechanism is given by the well known fact that renormalized particle-particle (κ) interactions can produce similar effects (i.e. the vanishing of the lowest pn-QRPA eigenvalue). The Hamiltonian corresponding to this case is $H = H_{11} + H_{\rm res} + H_{\rm int}$, where $H_{\rm res}$ is the above defined (8) symmetry restoring interaction and $H_{\rm int}$ is the Fermi separable force, with particle–hole and particle-particle terms included depending on the coupling constants χ and κ (22). Note that the repulsion induced by the particle–hole interaction (χ) is softened by the attractions induced by the symmetry restoring interaction (γ). The effect is particularly important for the eigenvalue of lowest energy and for the collective mode, which corresponds to the Isobaric Analog State (IAS). In the standard application of the pn-QRPA method the coupling constant χ is fixed in such a way that the position of the IAS is reproduced. If χ is varied, to reproduce a constant value of the energy of the IAS ($E_{\rm IAS}$) while changing γ, the collapse of the energy will be evident for values of $\gamma > \gamma_0$.

By increasing the ratio γ/γ_0 the value of κ which produces the collapse of the pn-QRPA spectrum decreases. It means that the renormalization of the P^-P^+ term of the Hamiltonian is limited by the break-down of the isospin symmetry at the level of the quasiparticle mean field.

From these results it is therefore concluded that the induced symmetry restoring interaction produces a strong renormalization of the κ-dependent interaction. This strong renormalization is by far more important than the one needed to produce the collapse of the pn-QRPA for at $\gamma = 0$. We can also say that the collapse of the pn-QRPA produced by particle-particle interactions is strongly dependent upon the spontaneous breaking of the isospin symmetry, which is forced by the BCS approximation. It was shown in [39] that the partial restoration of the symmetry can strongly reduce the value of κ for which the collapse is produced. It also means that the crossing of eigenvalues induced by κ [15] and the appearance of a zero eigenvalue associated to the symmetry ($\gamma = \gamma_0$) are different phenomena [6].

3 A comparative study on the validity of the Renormalized Random Phase Approximation

The Quasiparticle Random Phase Approximation (QRPA), introduced long ago by M. Baranger, has been extensively used to describe two-body correlations in open shell nuclei. The basic assumptions of the method are:

- the definition of a quasiparticle mean field to account for pairing correlations and
- the inclusion of two-body residual interactions between quasiparticles.

These assumptions have been tested rather successfully in dealing with the microscopic description of like-particle pair excitations in open shells. The first applications of the QRPA to describe unlike-particle pair excitations (proton-neutron pairs) were performed by Hableib and Sorensen [5]. The interest in the method was renewed by the study of the effects produced upon nuclear double-beta decay observables by renormalized two-particle (proton-neutron) interactions [8], [9].

Several attempts to prevent the so-called collapse of the proton-neutron (pn) QRPA approximation have been reported after the results of Vogel *et al.* [8] were published. For a review of some of these approaches see [10], [11]. Particularly, the method developed by Hara and applied to the Lipkin model by Catara *et al.* [12] has been extended to treat nuclear double-beta decay matrix elements by Toivanen and Suhonen [13]. Difficulties related to this approach have been reported in [14], [15].

The question about the validity of the RQRPA in realistic cases can not always be answered by a direct comparison with exact (shell model) results. However, the advantages and/or disadvantages of the RQRPA can be investigated in solvable models for which exact solutions are known.

In this section we shall compare RQRPA, QRPA and exact results for particle–hole monopole and quadrupole excitations and monopole pairing vibrational modes, both in normal and superfluid phases. Since some of the unknown nuclear-structure elements of the nuclear double-beta decay problem are not present in this case, i.e. the uncertainties associated with the treatment of ground state correlations induced by unlike proton–neutron pairs, we hope to extract more clear conclusions about the suitability of the RQRPA approach. We would like to concentrate on the ability of the RQRPA to describe correlations near the QRPA (RPA) phase transition.

The calculations were performed for schematic models where the single particle energy-spacing and the effective degeneracy of the single-particle levels are fixed. The coupling constants of the residual two-body interactions

were varied freely around critical values, i.e. the values of the RPA collapse, for each model Hamiltonian. Due to the schematic nature of the interactions self-consistent determinations of mean field properties were not implemented. Self-energy corrections to single-particle (quasiparticle-energies) and phonon-energies were not considered, either. The contributions of the interactions to RPA and Renormalized RPA processes were calculated at leading order in the expansion parameter Ω, which is the degeneracy of the single-particle levels. Other effects, like the re-arrangement of mean field values due to couplings with phonons and the renormalization of particle-vibration couplings due to phonon self-energies, were not considered here since they contribute at lower order in the $(1/\Omega)$ expansion. Limitations due to these approximations will be discussed with reference to schematic and realistic calculations reported in the literature.

Let us now introduce some basic notions about the RRPA (RQRPA) and RPA (QRPA) methods, as well as the definitions of the basis and generators of the algebra used to obtain exact solutions. Since most of the equations are well known, we are introducing them for the sake of completeness and long discussions on the formalism will be avoided. Each subsection of the present section will include the set of equations corresponding to a definite model situation and for it the exact solution will be shown together with the RPA (QRPA) and RRPA (RQRPA) solutions.

We shall now describe monopole pairing excitations (Sec. 3.1), monopole particle–hole excitations (Sec. 3.2), and quadrupole particle–hole excitations (Sec. 3.3). Each set of excitations is constructed by linear superpositions of like-quasiparticle pairs in the superfluid phase or like-particle–hole pairs in the normal phase. In addition, two-particle correlations will be treated to describe pairing vibrations.

3.1 Monopole Pair excitations

The correlated states of a system of N particles moving in two levels, each with a degeneracy 2Ω, are described by the Hamiltonian

$$H = \frac{\epsilon}{2}(N_2 - N_1) - G\Omega(A_2^\dagger + A_1^\dagger)(A_2 + A_1)\,, \tag{1}$$

where

$$A_\sigma^\dagger = \frac{1}{\sqrt{\Omega}} \sum_{m>0} a_{\sigma m}^\dagger \, a_{\overline{\sigma m}}^\dagger\,,$$
$$A_\sigma = (A_\sigma^\dagger)^\dagger\,,$$

$$N_\sigma = \sum_m a^\dagger_{\sigma m}\, a_{\sigma m}\ . \tag{2}$$

ϵ is the energy spacing between levels and the operator $a^\dagger_{\sigma m}$ ($a_{\sigma m}$) creates (annihilates) one-particle states denoted by $\{\sigma, m\}$, where $\sigma = 1, 2$ is the index associated to the upper (2) and lower (1) levels and $m = 1, \ldots, 2\Omega$ reads for the substates of each shell.

The operators $A^\dagger_\sigma, A_\sigma$ and N_σ obey the $SU(2)$ algebra. Exact eigenvalues and eigenvectors are obtained by a diagonalization in the basis

$$|k, n-k\rangle = \sqrt{\frac{(\Omega-k)!(\Omega-n+k)!}{k!\Omega!(n-k)!\Omega!}} A_1^{\dagger k} A_2^{\dagger (n-k)} |\phi\rangle\ , \tag{3}$$

where $A_1 \mid \phi\rangle = 0$, $0 \leq k \leq n$ and $2n = N$ is the number of fermions. The non-vanishing matrix elements of H are written

$$\begin{aligned}\langle k', n-k' \mid H \mid k, n-k\rangle &= \epsilon(n-2k)\delta_{k',k} \\ &-G(k(\Omega-k+1)+(n-k)(\Omega-n+k+1))\delta_{k',k} \\ &-G\sqrt{(k+1)(n-k)(\Omega-k)(\Omega-n+k+1)}\,\delta_{k',k+1} \\ &-G\sqrt{k(n-k+1)(\Omega-k+1)(\Omega-n+k)}\,\delta_{k',k-1}\ .\end{aligned} \tag{4}$$

3.1.1 The RPA Approximation for monopole pairing vibrations

The solutions given by the Random Phase Approximation (RPA), for the spectrum of H in the normal phase, can be written in terms of *addition* (a) and *removal* (r) one-phonon operators. For brevity, we shall introduce the equations for the addition modes (similar equations are obtained for the removal mode by replacing particles by holes). Thus

$$\Gamma^\dagger_a \ = \ X_a\, A^\dagger_2 - \ Y_a\, A^\dagger_1\ , \tag{5}$$

The harmonic version of H in the phonon basis is obtained from the commutator

$$[H, \Gamma^\dagger_a] \ = \ \omega_a\, \Gamma^\dagger_a\ , \tag{6}$$

with the eigenvalue $\omega_a \ = \ \epsilon\sqrt{1 - \frac{\Omega G}{(\epsilon/2)}}$.

The forward (X) and backward (Y)-going amplitudes are written as

$$X_a = \frac{\Lambda_a\sqrt{\Omega}}{\epsilon - \omega_a}\,, \qquad Y_a = \frac{\Lambda_a\sqrt{\Omega}}{\epsilon + \omega_a}\,, \tag{7}$$

with $\Lambda_a = \frac{G\sqrt{\Omega}}{\sqrt{\omega_a/\epsilon}}$.

3.1.2 The RPA Approximation in the Superfluid Phase (QRPA)

After applying the BCS transformation the Hamiltonian H, (1), is written as

$$\begin{aligned} H = & \sum_j E_j N_j - \frac{\Omega G}{2}\sum_{ij}(u_i^2u_j^2 + v_i^2v_j^2)(A_i^\dagger A_j + A_j^\dagger A_i) \\ & + \frac{\Omega G}{2}\sum_{ij}(u_i^2v_j^2 + u_j^2v_i^2)(A_i^\dagger A_j^\dagger + A_j A_i) \\ & + \sqrt{\Omega}G\sum_{ij} u_iv_i(u_j^2 - v_j^2)(A_j^\dagger N_i + N_i A_j) \\ & - G\sum_{ij} u_iv_iu_jv_j N_i N_j\,, \end{aligned} \tag{8}$$

The quasiparticle energies ($E_1 = E_2 = E$) and the pairing gap parameter (Δ) are given by

$$E = \Omega G \quad \text{and} \quad \Delta = \Omega G\sqrt{1 - (\frac{(\epsilon/2)}{\Omega G})^2}$$

respectively. The BCS occupation factors are defined by

$$u_1 = v_2 = u = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{(\epsilon/2)}{E}}\,, \qquad u_2 = v_1 = v = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{(\epsilon/2)}{E}}\,,$$

while the operators $A_\sigma^\dagger$, A_σ and N_σ of (8) are the same as in (2) but written in terms of quasiparticle creation and annihilation operators. These equations are correct at leading order in the parameter Ω. Self-insertions in the unperturbed quasiparticle term are neglected as well as the self-energy corrections to quasiparticle and phonon energies due to the quasiparticle-phonon coupling. These contributions are of lower order in Ω and for the schematic Hamiltonian (1) the omission of these corrections will not affect the QRPA or the RQRPA results in a significant manner.

3.1.3 *The Renormalized RPA*

The effect due to ground state correlations, upon the RPA (and QRPA) amplitudes and eigenvalues was studied by Hara and applied to the RPA treatment of monopole excitations by Catara *et al.* [12]. By scaling the pair creation operators $A^\dagger$, in the normal phase,

$$\tilde{A}^\dagger_\sigma \;=\; D_\sigma^{-1/2}\, A^\dagger_\sigma \;, \tag{9}$$

the Hamiltonian (1) is written

$$\begin{aligned} H = & \frac{\epsilon}{2}(N_2 - N_1) \\ & -G\Omega(\sqrt{D_2}\tilde{A}^+_2 + \sqrt{D_1}\tilde{A}^+_1)(\sqrt{D_2}\tilde{A}_2 + \sqrt{D_1}\tilde{A}_1) \;. \end{aligned} \tag{10}$$

The factors D should be determined self-consistently. The renormalized RPA equations, which define the scaled phonon creation operator (and similarly for the removal one-phonon operator)

$$\tilde{\Gamma}^\dagger_a \;=\; \tilde{X}_a\, \tilde{A}^\dagger_2 - \;\tilde{Y}_a\, \tilde{A}^\dagger_1 \;, \tag{11}$$

are written as

$$\begin{aligned} & \tilde{X}_a^2 - \tilde{Y}_a^2 = 1 \;, \\ & (\epsilon - \Omega G D_2 - \tilde{\omega}_a)\tilde{X}_a - \Omega G\sqrt{D_1 D_2}\;\tilde{Y}_a = 0 \;, \\ & -\Omega G\sqrt{D_1 D_2}\;\tilde{X}_a - (\epsilon - \Omega G D_1 + \tilde{\omega}_a)\tilde{Y}_a = 0 \;, \end{aligned} \tag{12}$$

with

$$D_1 = \frac{1}{1 + \tilde{Y}_a{}^2} \;, \qquad D_2 = \frac{1}{1 + \tilde{Y}_r{}^2} \;.$$

It implies that self-consistency can only be achieved by treating both the addition and removal modes simultaneously. The above structure, for the coefficients D, does not include an additional dependence on single-particle occupation numbers. These occupation numbers are calculated in the correlated ground state and they appear in the definition of the RPA (QRPA) metric. In the so-called self-consistent approach the right hand side of the QRPA (RPA) matrix equations includes a diagonal matrix with matrix elements defined by the ground state expectation values of the number operator, for each single

particle level. In this form, the local variation of the central single-particle density produced by phonon excitations is the source of RPA density fluctuations. However, and in the contexts of the present approximations, these corrections are also of lower order, as compared to the RPA (QRPA) leading order terms.

3.1.4 The renormalized QRPA approximation

In this approximation the phonon operator reads

$$\tilde{\Gamma}^{\dagger} = \tilde{X}(\tilde{A}_2^{\dagger} - \tilde{A}_1^{\dagger}) - \tilde{Y}(\tilde{A}_2 - \tilde{A}_1) \,, \tag{13}$$

After the renormalization given in (9) is applied we have obtained the following equations

$$\begin{aligned}
&2(\tilde{X}^2 - \tilde{Y}^2) = 1 \,, \\
&(2E - \frac{\epsilon^2}{E} D - \tilde{\omega})\tilde{X} - \frac{\epsilon^2}{E} D\tilde{Y} = 0 \,, \\
&-\frac{\epsilon^2}{E} D\tilde{X} - (2E - \frac{\epsilon^2}{E} D + \tilde{\omega})\tilde{Y} = 0 \,,
\end{aligned} \tag{14}$$

with

$$D = \frac{1}{1 + \tilde{Y}^2} \,. \tag{15}$$

As we have said above, these definitions are valid at leading order in Ω. Further corrections, to the single-quasiparticle occupation factors or to the quasiparticle and phonon energies, produce the mixing of orders.

3.1.5 The Dyson Boson Mapping

For the adopted single particle basis, the fermionic operators $N_\sigma, A_\sigma^{\dagger}, A_\sigma$ can be mapped onto a bosonic $(b_\sigma^{\dagger}, b_\sigma)$ space which preserves the original $SU(2)$ algebra, thus

$$\begin{aligned}
\sqrt{\Omega}\, A_2^{\dagger} &= b_2^{\dagger}(\Omega - b_2^{\dagger} b_2) \; (= \; \tilde{b}_2^{\dagger}), \\
\sqrt{\Omega}\, A_2 &= b_2, \\
N_2 &= b_2^{\dagger} b_2, \\
\sqrt{\Omega}\, A_1^{\dagger} &= b_1^{\dagger}(\Omega - b_1^{\dagger} b_1) \; (= \; \tilde{b}_1^{\dagger}),
\end{aligned}$$

$$\sqrt{\Omega}\,\overline{A}_1 = b_1,$$
$$N_1 = \Omega - b_1^\dagger b_1\ , \tag{16}$$

with $[b_j, b_i^\dagger] = \delta_{ij}$, and $\overline{A}_1^\dagger = A_1$.

In this space, bra- and ket-vectors are given by

$$|k, n-k\rangle = \sqrt{\frac{(\Omega-k)!}{k!\Omega!}\frac{(\Omega-n+k)!}{(n-k)!\Omega!}}\ \tilde{b}_2^{\dagger(n-k)}\tilde{b}_1^{\dagger(\Omega-k)}|0\rangle\ ,$$

$$\langle nk| = \sqrt{\frac{(\Omega-k)!}{k!\Omega!}\frac{(\Omega-n+k)!}{(n-k)!\Omega!}}\langle 0|\, b_2^{(n-k)}b_1^{(\Omega-k)}\ , \tag{17}$$

with $0 \le k \le n$.

At leading order in powers of Ω, the matrix elements of the Hamiltonian are written

$$\begin{aligned}\langle k', n-k' \mid H \mid k, n-k\rangle = &\Big(2\epsilon(n-2k)\\ &-G[k(\Omega-k+1)+(n-k)(\Omega-n+k+1)]\Big)\delta_{k',k}\\ &-G\Omega\sqrt{(k+1)(n-k)}(1-\frac{k}{2\Omega})(1-\frac{n-k-1}{2\Omega})\delta_{k',k+1}\\ &-G\Omega\sqrt{k(n-k+1)}(1-\frac{k-1}{2\Omega})(1-\frac{n-k}{2\Omega})\delta_{k',k-1}\ .\end{aligned} \tag{18}$$

3.2 Monopole Particle–Hole excitations

The Hamiltonian of the Lipkin–Meschkov–Glick (LMG) Model

$$H\ =\ \epsilon\, K_0 - \frac{1}{2}\, V\ (K_+^2 + K_-^2) \tag{19}$$

describes monopole particle–hole excitations in a two-level single-particle space, where

$$K_+ = \sum_m a_{2m}^\dagger a_{1m}\ ,$$
$$K_- = K_+^\dagger\ ,$$
$$K_0 = \frac{1}{2}\sum_m (a_{2m}^\dagger a_{2m} - a_{1m}^\dagger a_{1m})\ .$$

The exact solution of the model is obtained by considering the set of vectors

$$| k\rangle = \sqrt{\frac{(2\Omega - k)!}{k!(2\Omega)!}}\, K_+^k\, |\phi\rangle\,, \qquad 0 \le k \le 2\Omega\,, \tag{20}$$

$|\phi\rangle$ is a pure fermion state obeying $K_-|\phi\rangle = 0$. The non-vanishing matrix elements of H are

$$\begin{aligned}\langle k' \mid H \mid k\rangle \;=\; & \epsilon(k-\Omega)\,\delta_{k',k} \\ & -\frac{1}{2}V\sqrt{(k+1)(k+2)(2\Omega-k)(2\Omega-k-1)}\,\delta_{k',k+2} \\ & -\frac{1}{2}V\sqrt{k(k-1)(2\Omega-k+1)(2\Omega-k+2)}\,\delta_{k',k-2}\,.\end{aligned} \tag{21}$$

3.2.1 The RPA Approximation in the LMG Model

The one-phonon creation operator is defined by

$$\Gamma^\dagger = \sqrt{2\Omega}(X\, K_+ - \, Y\, K_-)\,. \tag{22}$$

and the corresponding RPA equation of motion yields the eigen-frequency $\omega = \epsilon\sqrt{1 - \left(\frac{2\Omega V}{\epsilon}\right)^2}$.

3.2.2 The renormalization of monopole particle–hole excitations

Introducing the scaling $\tilde{K}_+ = D^{-1/2}K_+$, the Hamiltonian (19) can be written as

$$H = \epsilon K_0 - \frac{1}{2}VD(\tilde{K}_+^2 + \tilde{K}_-^2) \tag{23}$$

with the corresponding renormalized one-phonon operator

$$\tilde{\Gamma}^\dagger \;=\; \sqrt{2\Omega}(\tilde{X}\, \tilde{K}_+ - \, \tilde{Y}\, \tilde{K}_-)\,, \tag{24}$$

The RPA equation of motion leads to the system of equations

$$\begin{aligned} & 2\Omega(\tilde{X}^2 - \tilde{Y}^2) = 1\,, \\ & (\epsilon - \tilde{\omega})\tilde{X} - 2\Omega VD\,\tilde{Y} = 0\,, \\ & -2\Omega VD\,\tilde{X} - (\epsilon + \tilde{\omega})\tilde{Y} = 0\,, \end{aligned} \tag{25}$$

where $D = 1/(1 + 2\tilde{Y}^2)$. Also here we have omitted the inclusion of monopole corrections to the single-particle energies, as well as additional corrections to the D factors given by higher powers of $\tilde{Y}$ in (21).

3.2.3 Boson Mapping of the LMG Hamiltonian

The fermionic operators $K_0, K_\pm$ can be transformed onto a boson $(b^\dagger, b)$ space which preserves the original $SU(2)$ algebra

$$\begin{aligned} K_+ &= b^\dagger(2\Omega - b^\dagger b) \ (= \ \tilde{b}^\dagger) \\ K_- &= b \\ K_0 &= b^\dagger b - \Omega \end{aligned} \tag{26}$$

with $[b, b^\dagger] = 1$.

In this space the bra- and ket-vectors are given by

$$\begin{aligned} |k\rangle &= \sqrt{\frac{(2\Omega - k)!}{k!(2\Omega)!}}\, \tilde{b}^{\dagger k}\, |0\rangle, \\ \langle k| &= \sqrt{\frac{(2\Omega - k)!}{k!(2\Omega)!}}\, \langle 0|\, b^k\,, \end{aligned} \tag{27}$$

with $0 \leq k \leq 2\Omega$.

The non-vanishing matrix elements of H are

$$\begin{aligned} \langle k' \mid H \mid k\rangle = {} & \epsilon k\ \delta_{k',k} \\ & -\frac{1}{2}V\sqrt{(k+1)(k+2)(2\Omega - k)(2\Omega - k - 1)}\ \delta_{k',k+2} \\ & -\frac{1}{2}V\sqrt{k(k-1)(2\Omega - k + 1)(2\Omega - k + 2)}\ \delta_{k',k-2}\,. \end{aligned} \tag{28}$$

This result is the same as the one obtained in the exact treatment of the model.

To leading order in Ω, the matrix elements of the Hamiltonian are given by

$$\begin{aligned} \langle k' \mid H \mid k\rangle \cong {} & \epsilon k\ \delta_{k',k} \\ & - V\Omega\sqrt{(k+1)(k+2)}\ \delta_{k',k+2} \\ & - V\Omega\sqrt{k(k-1)}\ \delta_{k',k-2}\,. \end{aligned} \tag{29}$$

To complete the study of the approximations described above in Secs. 3.1 and 3.1 we shall introduce the separable quadrupole–quadrupole interaction

$$
\begin{aligned}
H &= H_0 + H_{\text{int}} \,, \\
H_0 &= \sum_j E_j N_j \,, \\
H_{\text{int}} &= -g_2 \sum_\mu P^\dagger_{2\mu} P_{2\mu} + \kappa_2 \sum_\mu Q^\dagger_{2\mu} Q_{2\mu} \,,
\end{aligned}
\tag{30}
$$

where N is the quasiparticle-number operator and

$$
\begin{aligned}
P^\dagger_{2\mu} &= \sum_{j_1 j_2} p_2(j_1, j_2)(u_{j_1} u_{j_2}\, A^\dagger_{2\mu} - v_{j_1} v_{j_2}\, A_{\overline{2\mu}}) \,, \\
Q^\dagger_{2\mu} &= \sum_{j_1 j_2} t_2(j_1, j_2)(A^\dagger_{2\mu} + A_{\overline{2\mu}}) \,,
\end{aligned}
\tag{31}
$$

are two-quasiparticle components of the particle–particle and particle–hole quadrupole operator. The notation is given in [24]. It should be noted that quasiparticle states are taken as a crude representation of the mean field and that self-consistent corrections to the single particle (or single-quasiparticle) energies originated in the vacuum expectation value of the quadrupole operator are not considered. The lack of self-consistency introduced in this fashion will not affect the main trend of the renormalized results, as compared to the exact solutions. As in Secs. 3.1 and 3.2 the present discussion is restricted to the analysis of leading order effects.

Since an exact solution of the pairing plus quadrupole Hamiltonian in more than one shell cannot be formulated in terms of generators of a given algebra we shall restrict the analysis of the solutions to the one-shell limit. For this case (N active particles outside a core)

$$
H_{\text{int}} = H_{22} + H_{40} \,,
\tag{32}
$$

where H_{22} and H_{40} are the two and four quasiparticle terms of the Hamiltonian. Realistic values of the corresponding effective coupling constants g_{22} and g_{40}, defined by

$$
\begin{aligned}
g_{22} &= \kappa_2 \ |\, t_2 \,|^2 - \frac{1}{2}\, g_2 \ |\, p_2 \,|^2 \ (u^4 + v^4) \,, \\
g_{40} &= \kappa_2 \ |\, t_2 \,|^2 + g_2 \ |\, p_2 \,|^2 \ u^2 v^2 ,
\end{aligned}
\tag{33}
$$

are obtained by fixing $g_{22} = 0$ and consequently

$$\kappa_2 = \frac{g_2}{2R_0^4}\frac{u^4+v^4}{u^2v^2} . \tag{34}$$

With this set of parameters the Hamiltonian (30) reduces to the familiar form

$$H = \sum_j E_j N_j + g_{40}\frac{1}{2}\sum_\mu (A^\dagger_{2\mu}A^\dagger_{\overline{2\mu}} + A_{\overline{2\mu}}A_{2\mu}) . \tag{35}$$

which becomes linear in the quadrupole–phonon basis, by applying the QRPA transformations. The solution for the corresponding QRPA eigenvalue is written $\omega = 2E\sqrt{1-\left(\frac{g_{40}}{E}\right)^2}$.

3.3.1 RQRPA of the quadrupole-quadrupole interaction

Introducing the scaling (9) at the level of the two-quasiparticle operator $A^\dagger_{2\mu}$ the Hamiltonian of (35) is written

$$H = EN + g_{40}\frac{1}{2}\sum_\mu D_\mu^{1/2}(\tilde{A}^\dagger_{2\mu}\tilde{A}^\dagger_{\overline{2\mu}} + \tilde{A}_{\overline{2\mu}}\tilde{A}_{2\mu})D_\mu^{1/2} . \tag{36}$$

Thus, the RQRPA leads to the system of equations

$$\begin{aligned}
&\tilde{X}^2_{\mu n} - \tilde{Y}^2_{\mu n} = 1 ,\\
&(2E - \tilde{\omega}_n)\tilde{X}_{\mu n} - 2g_{40}\,\tilde{D}_\mu\tilde{Y}_{\mu n} = 0 ,\\
&2g_{40}\,\tilde{D}_\mu\tilde{X}_{\mu n} - (2E + \tilde{\omega}_n)\tilde{Y}_{\mu n} = 0 ,
\end{aligned} \tag{37}$$

for the renormalized amplitudes. The validity of these equations is restricted by the same considerations following (12), of Sec. 3.1.

3.3.2 Dyson Boson Mapping of the quadrupole Hamiltonian

It is performed by introducing the boson mapping

$$\begin{aligned}
A^\dagger_{2\mu} &= B^\dagger_{2\mu} - \sum_{\rho M_\rho \nu M_\nu \sigma M_\sigma} \Gamma^{\rho M_\rho \nu M_\nu}_{2\mu\sigma M_\sigma} B^\dagger_{\rho M_\rho} B^\dagger_{\nu M_\nu} B_{\sigma M_\sigma} (= \tilde{B}^\dagger_{2\mu}) ,\\
A_{2\mu} &= B_{2\mu} ,\\
a_{\mu\nu} &= \sum_{\rho M_\rho \sigma M_\sigma} \Gamma^{2\mu\rho M_\rho}_{2\mu\sigma M_\sigma} B^\dagger_{\rho M_\rho} B_{\sigma M_\sigma} ,
\end{aligned} \tag{38}$$

with $[B_{\rho M_\rho}, B^\dagger_{\sigma M_\sigma}] = \delta_{\rho\sigma}\delta_{M_\rho M_\sigma}$.

In the present case the underlying algebra is given by the $SO(2\Omega)$ representation defined by the commutators

$$\begin{aligned}
[A_{2\mu}, A^\dagger_{2\nu}] &= \delta_{\mu\nu} - 2a_{\mu\nu} \ , \\
[N, A^\dagger_{2\mu}] &= 2A^\dagger_{2\mu} \ , \\
[N, A_{2\mu}] &= -2A_{2\mu} \ ,
\end{aligned} \tag{39}$$

Vectors in this space are defined by

$$\begin{aligned}
|k(\mu)\rangle &= N(k(\mu))\tilde{B}^{\dagger k(\mu)}_{2\mu}|0\rangle \ , \\
\langle k(\mu)| &= N(k(\mu))\langle 0|B^{k(\mu)}_{2\mu} \ ,
\end{aligned} \tag{40}$$

the normalization factor $N(k(\mu))$ reads

$$N(k(\mu)) = \frac{1}{\sqrt{k(\mu)!\Omega(\Omega - f(\mu))\ldots(\Omega - (k(\mu)-1)f(\mu))}} \ , \tag{41}$$

with $f(\mu) = \frac{1}{2}\Gamma^{2\mu\,2\mu}_{2\mu\,2\mu}$ and $0 \leq k \leq \Omega$ ($k = \sum_\mu k(\mu)$).

The non-vanishing matrix elements of H are

$$\begin{aligned}
\langle k' \mid H \mid k\rangle &= 2Ek\delta_{k',k} \\
&-g_{40}\sum_\mu \sqrt{(k(\mu)+1)(k(-\mu)+1)} \\
&\quad \times \sqrt{(\Omega - k(\mu)f(\mu))(\Omega - k(-\mu)f(-\mu))} \\
&\quad \times \delta_{k'(\mu),k(\mu)+1}\delta_{k'(-\mu),k(-\mu)+1} \\
&-g_{40}\sum_\mu \sqrt{k(\mu)k(-\mu)} \\
&\quad \times \sqrt{(\Omega - (k(\mu)-1)f(\mu))(\Omega - (k(-\mu)-1)f(-\mu))} \\
&\quad \times \delta_{k'(\mu),k(\mu)-1}\delta_{k'(-\mu),k(-\mu)-1} \ .
\end{aligned} \tag{42}$$

We shall now present the results of the calculations, for the excitation energies and transition matrix elements of relevant operators, performed within the different approximations introduced above.

The exact results, for the case of monopole pairing vibrations, have been obtained by adopting the parametrization $\Omega = 20$, $N = 40$ and $\epsilon = 1$, for the shell degeneracy, the number of particles and the energy spacing between

levels, respectively. For the case of monopole particle–hole excitations, the adopted values are $\Omega = 4$, $N = 4$ and $\epsilon = 1$. The conclusions extracted from the present results will be limited by the schematic nature of the interactions which we have used to calculate RQRPA(RRPA) quantities. By the other hand the results have been consistently obtained at a given order in the expansion parameter Ω.

Before presenting our results we would like to comment, briefly, on some of the existing results of the RRPA (RQRPA). Let us concentrate first on the effects which we have consistently neglected in our calculations, like self-insertions and self-energy corrections to single-particle and phonon energies and the self-consistent coupling between the mean-field and the residual interactions. It is indeed true that the strength of the residual interaction should be fixed in a consistent manner. It will depend always on the single-particle basis and the renormalization of the couplings will also be a function of the number of single-particle states and their energies. In realistic situations single-particle properties, i.e. the sequence and density of levels around the Fermi surface, pairing properties and vibrational properties in double-even- and in double-odd-mass nuclei, depend upon each other. In principle, for a given two-body interaction and in a given single-particle basis, the HF mean field can be solved and the corresponding residual interactions can be treated consistently. In other approximations, the mean field is represented by empirical single-particle or quasi-particle levels taken from odd-even-mass and even-odd-mass nuclei and their mass differences and the strength of the residual interaction is fixed by reproducing the energy of the first excited state of the corresponding even-even- or odd-odd mass nuclei. This procedure leads to non-consistent couplings. In this respect, the removal of the spurious dipole state by selecting a suitable strength at the RPA level of approximation is a good example. For schematic Hamiltonians the contributions of single-particle and phonon degrees of freedom can be ordered in powers of the shell-degeneracy Ω. In this respect the RPA is a theory of small amplitude vibrations around the minima given by the single-particle mean field. Both the single particle or quasiparticle energies and the RPA (QRPA) energies are of order Ω and the couplings are of order $(1/\sqrt{\Omega})$. Self-insertions in the quasiparticle energies (i.e. gv^4) and the self-energy corrections due to the coupling to phonon states are of lower order. For large values of Ω these corrections can be neglected. Of course, for realistic situations they can affect the energy spacing between states above and below the Fermi surface.

For the cases which we have discussed, we have consistently worked at leading order in Ω to avoid the additional complications of mixing-orders and over-completeness. The inclusion of self-insertions and self-energy corrections, both to the fermions and phonons, will certainly change the point where the RPA collapses and the point where the renormalized RPA shows a departure respect to it. In schematic situations, like the situations discussed in the text, the

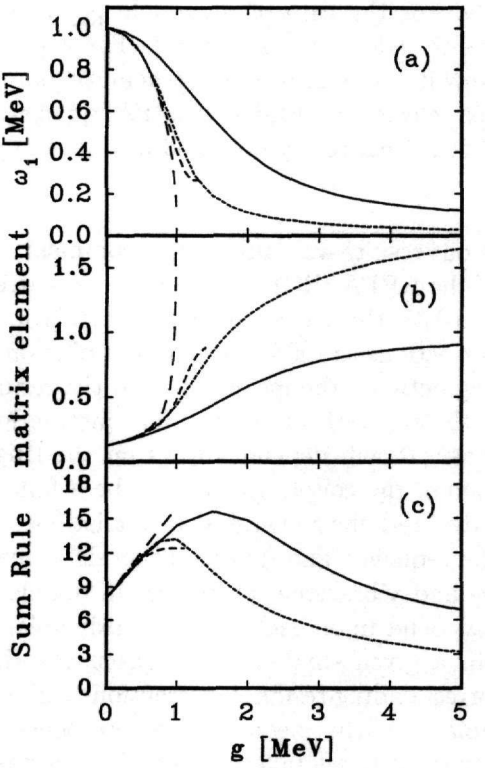


Fig. 1. Results corresponding to monopole particle–hole excitations in different approximations as a function of $g = 2\Omega V$. (**a**) The excitation energy of the first excited state (ω_1) shown in the inset. (**b**) The matrix elements of a monopole particle–hole operator [12] which induces transitions between the ground state and the first excited state. (**c**) The contributions to the monopole energy weighted sum-rule are shown in the inset. Representation of both the exact result and the result of Dyson's boson mapping to all order (*solid curve*); the RPA (QRPA) result (*long-dashed line*); the result based in Dyson's boson mapping at leading order in Ω (*short-dashed line*), and the results of the RRPA (RQRPA) (*dotted line*)

overall trend remains unaffected. From the published evidence we see that this kind of procedure, which for schematic models does not reproduce the exact solution either, does not work beyond the RPA-phase transition point, as shown by Delion *et al.*.

The results shown in Fig. 1, which correspond to monopole particle–hole excitation in the LMG model, do indeed reproduce the results of Catara *et al.* [12] concerning the value of the energy of the first excited state. This figure shows the characteristic collapse of the RPA eigenvalue at a certain critical value of the coupling constant $g = 2\Omega V$. By using the renormalized RPA method the collapse (i.e. $w \to 0$) is avoided, a trend which is also shown by the exact solution. The solution obtained by using Dyson's boson mapping at leading order is indeed very similar to the RRPA solution. However they

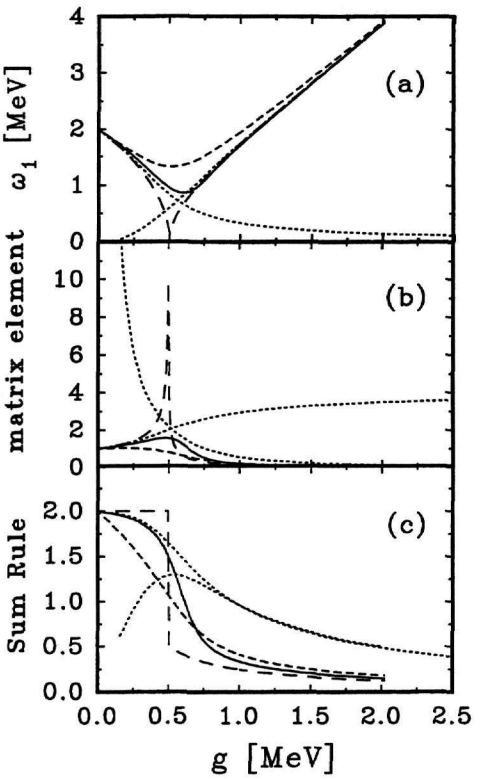


Fig. 2. Monopole pairing vibrations. Energy, matrix element of the two-particle transfer operator and contribution to the energy weighted sum rule, for transitions connecting the ground state and the first excited one-phonon state. (**a**) Excitation energy of the first excited state given by the different approximations, as a function of the coupling constant $g = \Omega G$. (**b**) Square matrix element of the two-particle transfer operator corresponding to the transition from the ground state to the first excited one. (**c**) Contribution to the energy weighted sum rule for the same transition. The results are displayed with curves that follow the convention used in Fig. 1

are still very different from the exact solution, for values of g larger than the critical value ($g_c = 1$) of Fig. 1. From these results one may be tempted to conclude that the RRPA method works fairly well, in spite of the fact that the RPA, RRPA and exact wave functions of the first excited state look very different, as it can be seen from the curves of Fig. 1c. One interesting feature of these curves is the fact that the Dyson boson mapping method at leading order and the renormalized RPA yield comparable results around g_c but both approximations differ strongly from the exact solution.

The behavior of the approximations in the vicinity of a phase transition is better illustrated, perhaps, by the case of monopole pairing vibrations. The well known separation between the normal and superfluid phases, as a function of the pairing coupling constant, is exhibited by the results for the energy of

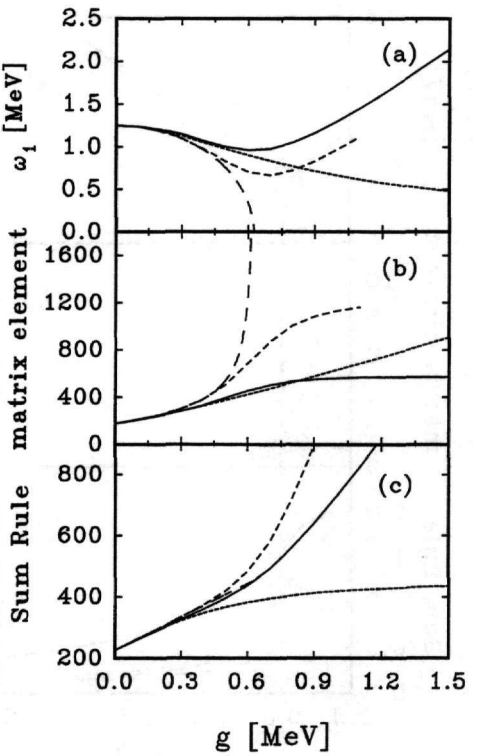


Fig. 3. Results corresponding to quadrupole Excitation Energy and Transition Matrix Elements. excitations in different approximations and for different values of $g = \Omega g_{40}$. (**a**) The excitation energies shown in the inset. (**b**) The transition matrix element of the quadrupole operator, for transitions between the ground state and the first excited state. (**c**) The contributions to the energy weighted sum-rule shown in the inset. The results are displayed with curves that follow the convention used in Fig. 1

the first excited state, as shown in Fig. 2a. The curves look very familiar and the discrepancy shown by the renormalized RPA(QRPA), as compared with the exact solutions, is evident. While the RPA(QRPA) produces a zero-energy eigenvalue at $g_c = 0.5$ (for this case $g = \Omega G$) the results of the renormalized approximations cross the critical point without vanishing. In fact, the results of Dyson Boson Mapping are better than other approximations, as compared with the exact results. Again in this case the strong differences between exact and renormalized wave functions at each side of the phase transition point are reflecting upon the dependence of the contributions to the sum rule (for transitions induced by the two-particle transfer operator, Fig. 2c).

Finally, the results corresponding to quadrupole excitations in a single shell, for the various methods discussed in the text, are shown in Fig. 3. The features of the solutions, for this case, are very much the same as those of the previous cases. The fact that the renormalized wave functions and the exact ones are

different is shown clearly by the curves of Fig. 3c.

The renormalization technique of Hara, as formulated by Catara *et al.* [12], was used to treat two-particle and particle–hole multipole excitations in schematic models. Although the eigenvalues obtained with the RRPA (RQRPA) are similar to the eigenvalues given by the exact and standard RPA(QRPA) methods, the renormalization procedure seemingly fails in reproducing the wave functions. The strong departure from the exact wave functions across and passing by the point where the RPA(QRPA) collapses, suggests that the renormalization method may not be able to account correctly for the correlations induced by the Hamiltonian. Obviously, these conclusions are limited by the very schematic structure of the models so far considered but it is worth to mention that the differences between the standard and renormalized RPA and exact solutions, may depend upon each Hamiltonian. While the agreement between exact and renormalized results, for the eigenvalues, is good for some cases, like the case considered in [12], they are not so good for other cases, like for monopole pairing and quadrupole excitations. Particularly, the crossing of a phase transition point, as shown in the case of pairing vibrational modes is a warning about the use of the renormalization technique in more realistic situations.

As seen from the results shown in this section it appears that the renormalization procedure seemingly works correctly in the regions where the naive RPA also works, that is in the region before the collapse, but it is also seen from the above results that a departure from the exact results is observed even in this region. As said before, the difference between exact and renormalized methods may also depend on the particular Hamiltonian used in the calculations. These features may also suggest that in actual applications of the renormalized RPA approach to realistic cases the question about its validity nearby a transition point still deserves to be discussed.

4 Collective description of nuclear double beta decay transitions

Here we shall adopt an alternative description based on the fact that the the zero-energy state is a consequence of the breakdown of the isospin symmetry implicit in the (separate) neutron (n) and proton (p) BCS solutions [2]. The procedure has been presented in [33], for the case of rotational degrees of freedom, and we shall briefly describe here its use for the case of isospin collective and intrinsic variables. Details can be found in [40].

The corresponding Hamiltonian is

$$H = \sum_v \left(e_v \tau_v \; - g_v S_v^+ S_v \right) - \frac{1}{2} g_\perp S_\perp^+ S_\perp \tag{1}$$

where

$$S_v^+ = \sum_{m>0} c_{vm}^+ c_{v\overline{m}}^+ \; ; \qquad S_\perp^+ = \sum_{m>0} (c_{pm}^+ c_{n\overline{m}}^+ + c_{nm}^+ c_{p\overline{m}}^+) \; ,$$

$$\tau_A = \frac{1}{2}(\tau_p + \tau_n) \; ; \quad \tau_0 = \frac{1}{2}(\tau_p - \tau_n) \; ; \quad [\tau_{\overline{1}}, \tau_1] = \tau_0 \; ,$$

$$T_A = \frac{1}{2}(T_p + T_n) \; ; \quad T_0 = \frac{1}{2}(T_p - T_n) \; ; \quad [T_{\overline{1}}, T_1] = -T_0 \; .$$

The introduction of collective degrees of freedom is compensated through the appearance of the constraints

$$\tau_z - T_z = 0 \; ; \qquad (z = n, p, \pm 1) \; , \tag{2}$$

which express the fact that we can rotate the intrinsic system in one direction or the body in the opposite one without altering the physical situation [33]. Physical states should be annihilated by the four constraints and physical operators should commute with them.

The collective Hilbert space appropriate for an isospin conserving pairing interaction was originally introduced in [2], [34], [35]. The states may be labeled by the four quantum numbers $|T_A, T, m, k\rangle$, where T_A is the total number of pairs of particles. Here we substitute M, T_0 (the isospin projections in the laboratory and intrinsic frames) by the quantum numbers $m \equiv \frac{1}{2}(T + M)$ and $k \equiv \frac{1}{2}(T + T_0)$, respectively. We focus on states such that $m \ll T$ and $k = 0$.

Non-physical violations of the isospin symmetry are allowed in the intrinsic frame. Such frame may be defined by the condition $\overline{S}_\perp = 0$, where the bar denotes the g.s. expectation value [35]. This condition is precisely satisfied by performing the usual separate Bogoliubov transformation for protons and neutrons. The rotations in isospace and gauge space restore the symmetries which are present in the laboratory frame. Thus the *np*-pairing becomes effectively incorporated, as well as the pairing between identical particles.

However, as different from previous cases where collective coordinates have been used, we are dealing here with an interaction which does not conserve isospin. Namely, the Hamiltonian is not generally an isoscalar. As in most collective treatments, physical isotensor operators must be transformed from the laboratory frame to the intrinsic frame. In the case of the single-particle and pairing Hamiltonian this procedure yields:

$$
\begin{aligned}
H^{(\mathrm{lab})}_{\mathrm{sp}} &= e_A \tau_A + e_0 (D^1_{00}\tau_0 + D^1_{01}\tau_1 + D^1_{0\bar{1}}\tau_{\bar{1}}) \,, \\
H^{(\mathrm{lab})}_{\mathrm{pair},0} &= -g_0 (S^+_p S_p + S^+_n S_n + \frac{1}{2} S^+_\perp S_\perp) \,, \\
H^{(\mathrm{lab})}_{\mathrm{pair},1} &= -g_1 \Big[D^1_{00} (S^+_p S_p - S^+_n S_n) \\
&\qquad - \frac{D^1_{01}}{\sqrt{2}} (S^+_p S_\perp + S^+_\perp S_n) + \frac{D^1_{0\bar{1}}}{\sqrt{2}} (S^+_n S_\perp + S^+_\perp S_p) \Big] \,, \\
H^{(\mathrm{lab})}_{\mathrm{pair},2} &= -g_2 \Big\{ D^2_{00} (S^+_p S_p + S^+_n S_n - S^+_\perp S_\perp) \\
&\qquad - \sqrt{\frac{3}{2}} [D^2_{01} (S^+_\perp S_n - S^+_p S_\perp) + D^2_{0\bar{1}} (S^+_\perp S_p - S^+_n S_\perp)] \\
&\qquad + \sqrt{6} (D^2_{02} S^+_p S_n + D^2_{0\bar{2}} S^+_n S_p) \Big\} \,. \qquad (3)
\end{aligned}
$$

Here the subindices 0,1,2 on the l.h.s. denote isoscalar, isovector and isoquadrupole components, and

$$
\begin{aligned}
& e_A = e_p + e_n \,, \qquad e_0 = e_p - e_n \,, \\
& g_0 = \frac{g_p + g_n + g_\perp}{3} \,, \quad g_1 = \frac{g_p - g_n}{2} \,, \quad g_2 = \frac{g_p + g_n - 2g_\perp}{6} \,.
\end{aligned}
$$

It is easy to verify that the four components of this Hamiltonian commute with the constraints and are therefore physical operators.

Up to now the Hamiltonian together with the constraints constitute an exact reformulation of the original problem, since the introduction of additional collective coordinates is compensated by the presence of the constraints. Systems of this type can be treated in a perturbative way within an expansion given by the inverse order parameter $1/\overline{S}_v$, for instance through the BRST procedure [36], as applied to many-body problems in [33], and to the particular case of high angular momentum in [37]. There is, however, a new feature in the present case, namely the presence of the rotational matrices $D^\lambda_{\mu\nu}$ in the Hamiltonian. This extra complication can be overcome by means of Marshalek's generalization of the Holstein-Primakoff representation [38], which is amenable to an expansion in powers of T^{-1}. In what follows we will keep only the lowest order terms in such an expansion, assuming $O(\overline{S}_v) = T$ and $O(g_\nu) = T^{-1}$.

Such terms include the two (pp and nn) pairing Hamiltonians in a single j-shell $e_v \tau_v - g_v S^+_v S_v$, which are separately treated within the BCS approximation. In doing so, Lagrange multiplier terms $-\lambda_v(\tau_v - T_v)$ have to be added. This treatment yields the independent quasi-particle energies $E_v = \frac{1}{2}\Omega g_v$, where Ω is half the value of the shell degeneracy.

The spectrum of the system is ordered into collective bands, each one carrying as quantum numbers the total number of particles and the isospin ($T \leq T_A$).

The properties of these bands are obtained by adding the remaining leading order terms in to the independent np quasi-particle energy terms. To leading order in T^{-1}

$$
\begin{aligned}
&H^{(\mathrm{lab})}_{(\mathrm{sp+pair})} = \overline{H} + \omega_d d^+ d + H_2 \,, \\
&\overline{H} = \sum_v \left(e_v \overline{\tau}_v - g_v \overline{S}^2_v \right) \,, \\
&\omega_d = e_0 + \frac{g_1}{T}(\overline{S}^2_p - \overline{S}^2_n) + \frac{3g_2}{T}(\overline{S}^2_p + \overline{S}^2_n) \,, \\
&\langle T_A, T-2, m-2 \,|\, H_2 \,|\, T_A, T, m \rangle = -\frac{3g_2}{T}\overline{S}_p \overline{S}_n \sqrt{m(m-1)}) \,,
\end{aligned}
\tag{4}
$$

plus null terms. The boson creation operator $d^\dagger$ increases in one unit the value of m [38].

The energy of the band head is given by the BCS expectation value $\overline{H}$. The different members of each band are labeled by the quantum number m and are separated by the distance ω_d, which includes the difference between the proton and the neutron single-particle energies e_0. Our strategy has been to restore the number of particles T_A and the isospin T as good quantum numbers and, within such a basis, to construct the interband interaction H_2, which allows for the possibility of double-beta decay. In such a way we have been able to disentangle the physical isospin violations from the unphysical ones.

Both within the simple model or in the realistic case, the $\tau_{\pm 1}$ mode disappears from the final physical Hamiltonian to become part of the constraints. This is precisely the (unrenormalizable) phonon that yields a zero frequency root for isoscalar Hamiltonians within a naive RPA [32]. From the practical point of view it is as if this (unphysical) RPA boson becomes substituted by the collective boson d^+, d, which is well behaved in the limit of zero frequency. In realistic cases this structure is also maintained, but superimposed to the excitations of the other (physical) RPA modes. This substitution also becomes apparent in the expression for the strong current that appears in the weak Hamiltonian, which is proportional to the isospin operator, namely

$$
\begin{aligned}
\beta_- &= -\sqrt{2}\tau^{(\mathrm{lab})}_1 = -\sqrt{2}(D^1_{11}\tau_1 + D^1_{10}\tau_0 + D^1_{1\overline{1}}) \\
&\approx \sqrt{2T}\, d^+ + \text{null operator} \,.
\end{aligned}
\tag{5}
$$

From the point of view of the expansion in powers of T^{-1}, the interband interaction H_2 is of the same order (O(1)) as the distance between the states that are mixed by it. Nevertheless, in the following we continue applying perturbation theory by requiring that $|g_2| < g_v$.

Let us proceed now with the discussion of some calculations. The results are presented in [40]. As shown there, we predict the exact results for $g_2 = 0$ and very satisfactory ones for the other values, in spite of the fact that for these results we have neglected the interband interaction. The matrix element of double beta decay transitions, which for the present case correspond to pure Fermi transitions (cf. [32]), is proportional to the product of the two matrix elements

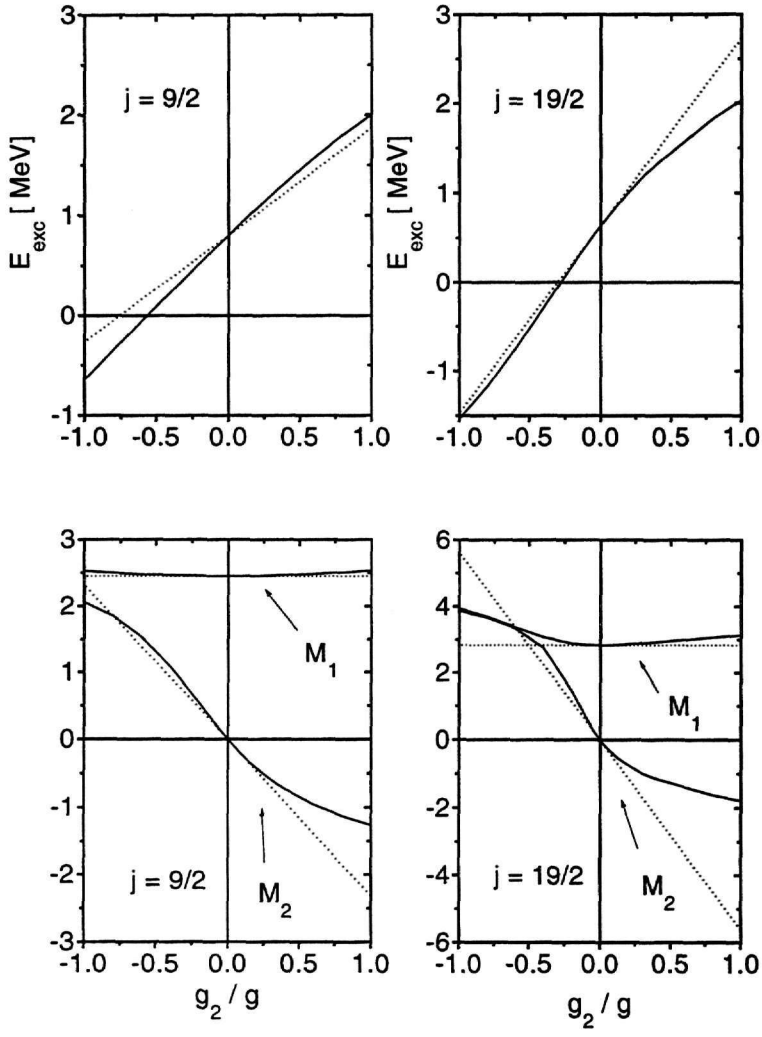


Fig. 1. Excitation Energy and Transition Matrix Elements. Exact (*solid lines*) and perturbative (*dotted lines*) results for the excitation energy (*upper boxes*) and transition matrix elements M_1 and M_2 (*lower boxes*) corresponding to the two different sets of parameters ($j = 9/2$ and $j = 19/2$). The results are taken from [40]

$$M_2 = \langle T_A, T-2, 0|\beta_-|T_A, T, 1\rangle$$
$$\approx -\frac{2\sqrt{T}\langle T_A, T-2, 0|H_2|T_A, T, 2\rangle}{\overline{H}(T_A, T, 2) - \overline{H}(T_A, T-2, 0)} \,. \tag{6}$$

These matrix elements are displayed in the lower boxes of Fig. 1. The expression for the interband matrix element does not distinguish whether the r.h.s. should be calculated for the initial or the final value of T, since it is valid for Tg^21. Therefore, the effective interband matrix element has been chosen as the ge ed bands.

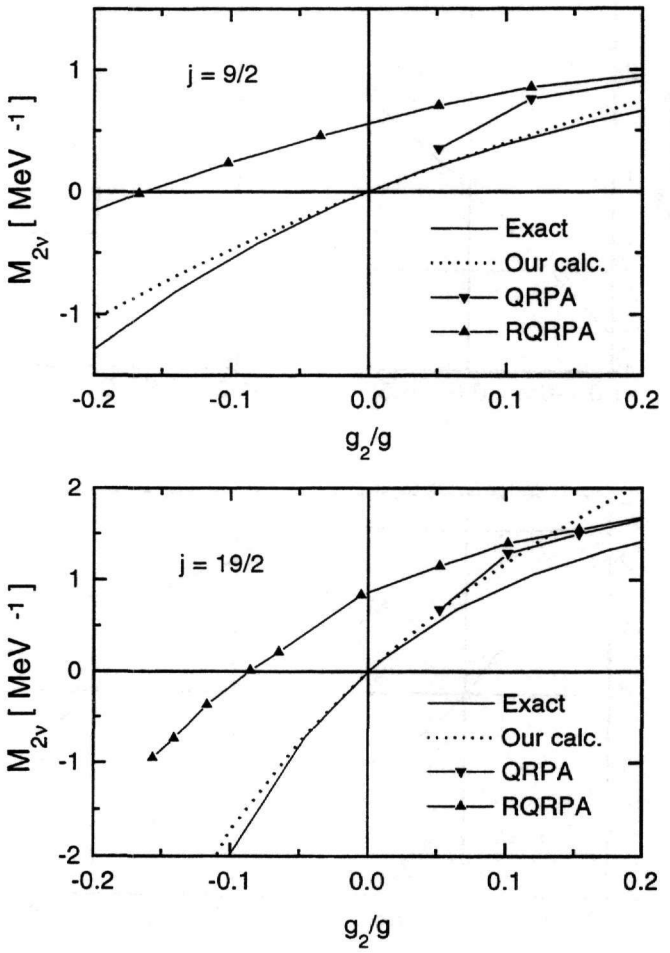


Fig. 2. Matrix elements for Fermi double beta decay transitions calculated in the Quasiparticle Random Phase Approximation (QRPA), the Renormalized Quasiparticle Random Phase Approximation (RQRPA) and in the perturbative treatment presented in [40]. Exact results are shown also and the calculations have been performed in one single j-shell

Figure 2 displays Fermi double beta decay matrix elements, corresponding to transitions from the initial to the final ground states. It has been calculated using the expression

$$M_{2\nu} = \frac{M_1 M_2}{\omega_d + \Delta} , \tag{7}$$

where the energy released Δ is taken to be 0.5 MeV, as in [32]. In addition to the exact and perturbative values of these matrix elements, we have included in Fig. 2 the results obtained by using some other approximations. The exact result shows the suppression of the matrix element around the point where the strength of the *np* symmetry breaking interaction approaches the value of the fully symmetric interaction. This result is reproduced both in the naive QRPA and in the perturbative approach. The other approximation badly misses this cancellation. A detailed comparison between the results of exact, naive QRPA and renormalized QRPA (RQRPA) calculations can be found in [32]. It is worth to note that in the perturbative approach the corresponding sum rule (Ikeda's sum rule) is exactly observed. This is not the case of other approaches, as the RQRPA. The perturbative approach, as seen in Figs. 1 and 2, not only reproduces exact results very satisfactorily but it also gives some insight about the mechanism responsible for the suppression of the matrix elements. As found in the calculations, the value of the matrix element M_2 depends critically on the strength of the physical symmetry breaking term H_2. On the other hand, the values of M_1 are not very much dependent on this interaction. Finally, it should be observed that the point where the excitation energy vanishes and the point where the symmetry is completely restored are different (Fig. 1). This result, also obtained in the exact diagonalization of the full Hamiltonian, cannot be reproduced by other means as shown in [32].

In conclusion, it is found that a correct treatment of collective effects induced by isospin dependent residual interactions in a superfluid system is feasible: physical effects due to the isospin symmetry-breaking terms in the Hamiltonian are obtained even in the presence of the BCS mean field built upon separate proton and neutron pairing interactions. The interplay of intrinsic and collective coordinates guarantees that the isospin symmetry is restored and that spurious contributions to the wave functions are decoupled from physical ones. Particularly, the problem of the instabilities found in the standard *np* QRPA are avoided by the explicit elimination of the zero frequency mode from the physical spectrum (but keeping it in the perturbative expansion). The appearance of this mode cannot be avoided by the inclusion of higher order terms in the QRPA expansion or by any other ad-hoc renormalization procedure, like the RQRPA, once the BCS procedure is adopted for the separate treatment of *pp*- and *nn*-pairing correlations [32].

Acknowledgments

This work has been partially supported by the CONICET and the project PMT-PICT0079 (Argentina). The results presented in Sec. 3 of this lecture have been obtained in collaboration with D.R. Bes and N.N. Scoccola (Lab. Tandar, CNEA). Section 2 is based on our common work with J. Hirsch and P. Hess (CINVESTAV and UNAM, Mexico) and M. Reboiro (UNLP). The hospitality received while in Ioannina is gratefully acknowledged.

References

[1] M. Baranger, Phys. Rev. **120**, 957 (1960).

[2] A. Bohr and B. Mottelson, *Nuclear Structure*, vol. 2 (Benjamin, Reading, MA, 1975).

[3] P. Ring and P. Shuck, *The nuclear many-body problem*, (Springer Verlag, N.Y. 1980).

[4] D.J. Rowe, *Nuclear Collective Models*, (Methuen and Co. Ltd, 1970).

[5] J.A. Halbleib and R.A. Sorensen, Nucl. Phys. **A98**, 542 (1967).

[6] A.M. Lane and J. Martorell, Ann. Phys. (NY) **129**, 273 (1980).

[7] E.R. Marshalek and J. Weneser, Ann. Phy. (NY) **53**, 569 (1969).

[8] P. Vogel and M.R. Zirnbauer, Phys. Rev. Lett. **57**, 3148 (1986).

[9] O. Civitarese, A. Faessler, and T. Tomoda, Phys. Lett. **B194**, 11 (1987).

[10] M.K. Moe and P. Vogel, Ann. Rev. Nucl. Part. Sci. **44**, 247 (1994).

[11] J. Suhonen and O. Civitarese, Phys. Rep. **300**, 123 (1998).

[12] F. Catara, N. Dinh Dang and M. Sambataro, Nucl. Phys. **A579**, 1 (1994).

[13] J. Toivanen and J. Suhonen, Phys. Rev. Lett. **75**, 410 (1995).

[14] J. Engel, S. Pittel, M. Stoitsov, P. Vogel and J. Dukelsky, Phys. Rev. **C55**, 1781 (1997).

[15] J. Hirsch, P. Hess and O. Civitarese, Phys. Rev. **C56**, 199 (1997).

[16] J. Hirsch, P. Hess and O. Civitarese, Phys. Lett. **B412**, 1 (1997).

[17] D. Bes, O. Civitarese, E. Maqueda and N. Scoccola, to be published.

[18] O. Civitarese and M. Reboiro, Phys. Rev. **C56**, 1179 (1997).

[19] O. Civitarese, M. Reboiro and P. Vogel, Phys. Rev. **C56**, 1840 (1997).

[20] N.I. Pyatov and M.I. Chernei, Sov. J. Nucl. Phys. **16**, 514 (1973) [Yad. Fiz. **16**, 931 (1972)];
M.I. Banzat and N.I. Pyatov, Sov. J. Nucl. Phys. **21**, 365 (1975) [Yad. Fiz. **21**, 708 (1975)].

[21] P.M. Dirac, *Lectures in Quantum Mechanics*, (Yeshiva University, New York, 1964).

[22] O. Civitarese and M.C. Licciardo, Phys. Rev. **C38**, 967 (1988).

[23] O. Civitarese and M.C. Licciardo, Phys. Rev. **C41**, 1778 (1990).

[24] O. Civitarese, A. Faessler and M.C. Licciardo, Nucl. Phys. **A542**, 221 (1992).

[25] H. Sakamoto and T. Kishimoto, Nucl. Phys. **A528**, 73 (1991).

[26] A. Bohr and B. Mottelson, *Nuclear Structure*, vol. 1 (Benjamin, Reading, MA, 1971)

[27] V.A. Kuz'min and V.G. Soloviev, Nucl. Phys. **A486**, 118 (1988).

[28] O. Civitarese and J. Suhonen, Nucl. Phys. **A578**, 62 (1994).

[29] W.C. Haxton and G.J. Stephenson, Jr. Prog. Part. Nucl. Phys. **12**, 409 (1984).

[30] F. Bohm and P. Vogel, *Physics of Massive Neutrinos*, (Cambridge Univ. Press, 2nd edition, Cambridge 1992).

[31] E. Caurier, F. Nowacki, A. Poves and J. Retamosa, Phys. Rev. Lett. **77**, 1954 (1996).

[32] J. Hirsch, P.O. Hess and O. Civitarese, Phys. Rev. **C56**, 199 (1997).

[33] D.R. Bes and J. Kurchan, *The treatment of collective coordinates in many-body systems*, (World Scientific, Singapore, 1990).

[34] J. Ginocchio and J. Weneser, Phys. Rev. **170**, 859 (1968).

[35] G.G. Dussel *et al.*, Nucl. Phys. **A 175**, 513 (1971);
G.G. Dussel, R.P.J. Perazzo and D.R. Bes, Nucl. Phys. **A 183**, 298 (1972);
D.R. Bes *et al.*, Nucl. Phys. **A217**, 93 (1973).

[36] C. Becchi, A. Rouet and S. Stora, Phys. Lett. **B52** , 344 (1974).

[37] J. Kurchan, D.R. Bes and S. Cruz Barrios, Nucl. Phys. **A509**, 306 (1990);
J.P. Garrahan and D.R. Bes, Nucl. Phys. **A573**, 448 (1994).

[38] E.R. Marshalek, Phys. Rev. **C11**, 1426 (1975).

[39] O. Civitarese, P.H. Hess, J.G. Hirsch and M. Reboiro, Phys. Rev. **C59**, 194 (1999).

[40] D.R. Bes, O. Civitarese and N.N. Scoccola, Phys. Lett. **B446**, 93 (1999).