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# The quark-meson model and the phase diagram of QCD

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I discuss the QCD phase diagram in the context of the linear quark-meson model with two flavours, using the exact renormalization group. I first give a pedagogical derivation of the qualitative features of the phase diagram based on mean field theory. Then I summarize how the the universality classes of the second-order phase transitions can be determined through the exact renormalization group. For non-zero quark masses I explain how the universal equation of state of the Ising universality class can be used in order to describe the physical behaviour near the critical point. The effective exponents that parametrize the growth of physical quantities, such as the correlation length, are given by combinations of the critical exponents of the Ising class that depend on the path along which the critical point is approached. In general the critical region, in which such quantities become large, is smaller than naively expected.

## 1. INTRODUCTION

The most prominent feature of the phase diagram of QCD at non-zero temperature and baryonic density is the critical point that marks the end of the line of first-order phase transitions. (For a review see ref. [1].) Its exact location and the size of the critical region around it determine its relevance for the heavy-ion collision experiments at RHIC and LHC.

In order to discuss the phase diagram one has to employ an appropriate order parameter. The usual choice is the quark-antiquark condensate that is associated with the spontaneous breaking of the chiral symmetry of the QCD Lagrangian. For two flavours an equivalent description uses as effective degrees of freedom four mesonic scalar fields (the  $\sigma$ -field and the three pions  $\pi_i$ ), arranged in a  $2 \times 2$  matrix

$$\Phi = \frac{1}{2} (\sigma + i\vec{\pi} \cdot \vec{\tau}), \quad (1)$$

where  $\tau_i$  ( $i = 1, 2, 3$ ) denote the Pauli matrices. The various interactions must be invariant under a global  $SU(2)_L \times SU(2)_R$  symmetry acting on  $\Phi$ . If the  $\sigma$ -field (that corresponds to a condensate  $\bar{u}u + \bar{d}d$ ) develops an expectation value the symmetry is broken down to  $SU(2)_{L+R}$ . The explicit breaking of the chiral symmetry through the current quark mass can be incorporated as well. It corresponds to the interaction with an external source through a term  $-j\sigma$  in the Lagrangian, with  $j$  proportional to the mass.

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## 2. THE MODEL

The full QCD dynamics cannot be described by an effective Lagrangian involving only mesonic fields. More complicated effective descriptions, such as the linear quark-meson model, have been used for the discussion of the phase diagram for low chemical potential [2–4]. The Lagrangian density has the form

$$\begin{aligned} \mathcal{L} = & i\bar{\psi}^a(\gamma^\mu\partial_\mu + \mu\gamma^0)\psi_a + h\bar{\psi}^a\left[\frac{1+\gamma^5}{2}\Phi_a{}^b - \frac{1-\gamma^5}{2}(\Phi^\dagger)_a{}^b\right]\psi_b \\ & + \partial_\mu\Phi_{ab}^*\partial^\mu\Phi^{ab} + U(\rho) - \frac{1}{2}\left(\Phi_{ab}^*j^{ab} + j_{ab}^*\Phi^{ab}\right). \end{aligned} \quad (2)$$

The fermionic field  $\psi^a$  includes two flavours ( $a = 1, 2$ ) corresponding to the up and down quarks. The non-zero temperature effects can be taken into account by restricting the time coordinate within the finite interval  $[0, 1/T]$  in Euclidean space. The scalar (fermionic) fields obey periodic (anti-periodic) boundary conditions. We shall also consider a non-zero chemical potential, associated to the quark number density  $\psi_a^\dagger\psi^a$  that is proportional to the baryon number. We have included a Yukawa coupling  $h$  between the constituent quarks and the mesonic field  $\Phi$ . The constituent quark mass is generated through the expectation value of  $\Phi$ . The last term in eq. (2) accounts for the explicit breaking of the chiral symmetry through the current quark masses. One expects  $j \sim M = \text{diag}(m_u, m_d)$ . We assume equal masses  $\hat{m}$  for the two light quarks so that  $j^a{}_b \sim \hat{m}\delta^a{}_b$ . The determination of the proportionality constant requires the embedding of the quark-meson model in a more fundamental framework [4–6].

The behaviour near second order phase transitions is determined by the dynamics at energy scales below the temperature  $T$ . At these scales the theory becomes effectively three-dimensional, as the Euclidean time coordinate is compactified at non-zero temperature. The basic structure of the phase diagram can be obtained if we assume that the potential takes the form

$$V(\rho) = m^2\rho + \frac{1}{2}\lambda\rho^2 + \frac{1}{3}g\rho^3, \quad (3)$$

with  $\rho = \text{Tr}(\Phi^\dagger\Phi) = (\sigma^2 + \pi_i\pi^i)/2$ . The couplings  $m^2$ ,  $\lambda$ ,  $g$  are considered functions of the temperature  $T$  and the chemical potential  $\mu$ . We assume that  $g$  is always positive, so that the potential is bounded from below. The effect of a current quark mass can be taken into account by adding a term  $-j\sigma$ , with  $j$  a (linear) function of the mass.

Within the quark meson model, if certain approximations are made, the relevant potential can be cast in the form of eq. (3) with [7,8]

$$m^2 = -\lambda_{0R}\rho_{0R} + \left[\frac{\lambda_{0R}}{4}\left(1 - \frac{3}{\pi^{3/2}}\right) + \frac{h^2N_c}{12}\right]T^2 + \frac{h^2N_c}{4\pi^2}\mu^2 \quad (4)$$

$$\begin{aligned} \lambda = & \lambda_{0R} + \frac{h^4N_c}{16\pi^2}\left[-1 + 2\gamma_e + \ln\left(\frac{\tilde{\Lambda}^2}{4T^2}\right) + 2\text{Li}^{(1,0)}\left(0, -\exp\left(\frac{\mu}{T}\right)\right)\right. \\ & \left. + 2\text{Li}^{(1,0)}\left(0, -\exp\left(-\frac{\mu}{T}\right)\right)\right]. \end{aligned} \quad (5)$$

Here  $\rho_{0R}$  is the renormalized expectation value for the mesonic field at zero temperature and chemical potential, while  $\lambda_{0R}$  is the renormalized mesonic quartic coupling. The parameter  $N_c = 3$  corresponds to the number of colours and  $\gamma_e \simeq 0.5772$  is the Euler-Mascheroni constant.  $\text{Li}^{(l_1, l_2)}(n, x)$  denotes the  $l_1$ -th and  $l_2$ -th partial derivative of the polylogarithmic function  $\text{Li}(n, x)$  with respect to  $n$  and  $x$  respectively. The function  $\text{Li}^{(1,0)}(0, -\exp(x)) + \text{Li}^{(1,0)}(0, -\exp(-x))$  is monotonically decreasing, and takes the value  $-\ln(\pi/2)$  for  $x = 0$ . Even if the mass term is negative at  $T = \mu = 0$ , it can become positive for sufficiently large  $T$  or  $\mu$ . For constant  $T$ , the quartic coupling is a decreasing function of  $\mu$ . For a certain value of  $\mu$  we can have  $\lambda = 0$ .

### 3. MEAN-FIELD THEORY

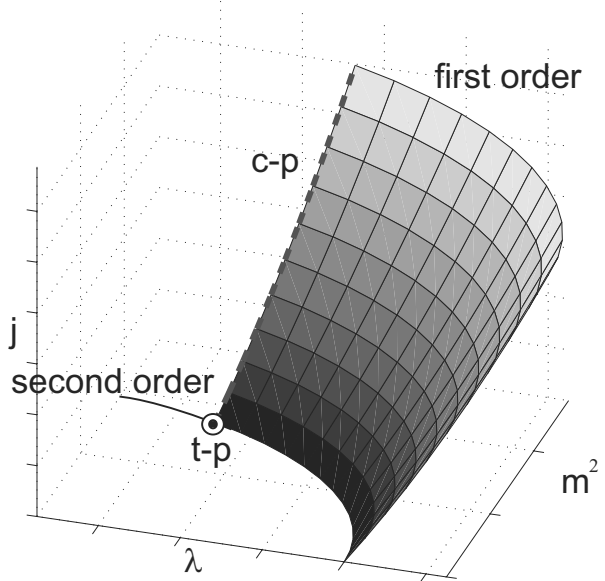
It is instructive to discuss the phase diagram of our model neglecting the field fluctuations. We allow for variations of  $m^2$  and  $\lambda$ , while we assume that the coupling  $g$  remains constant. This is a good approximation of the behaviour of the potential studied in ref. [7,8].

We begin by studying the phase diagram in the absence of an external source ( $j = 0$ ). Let us consider first the case  $\lambda > 0$ . For  $m^2 > 0$  the minimum of the potential is located at  $\sigma = \pi_i = 0$  and the system is invariant under an  $O(4)$  symmetry. For  $m^2 < 0$  the minimum moves away from the origin. Without loss of generality we take it along the  $\sigma$  axis. The symmetry is broken down to  $O(3)$ . The  $\pi$ 's, which play the role of the Goldstone fields, become massless at the minimum. If  $m^2(T, \mu)$  has a zero at a certain value  $T = T_{cr}$ , while  $\lambda(T_{cr}, \mu) > 0$ , the system undergoes a second-order phase transition at this point. The minimum of the potential behaves as  $\sigma_0 \sim |T - T_{cr}|^{1/2}$  slightly below the critical temperature. The critical exponent  $\beta$  takes the mean-field value  $\beta = 1/2$ .

Let us consider now the case  $\lambda < 0$ . It is easy to check that, as  $m^2$  increases from negative to positive values (through an increase of  $T$  for example), the system undergoes a first-order phase transition. For  $|\lambda|$  approaching zero the phase transition becomes progressively weaker: The discontinuity in the order parameter (the value of  $\sigma$  at the minimum) approaches zero. For  $\lambda = 0$  the phase transition becomes second order. The minimum of the potential behaves as  $\sigma_0 \sim |T - T_{cr}|^{1/4}$ . The critical exponent  $\beta$  takes the value  $\beta = 1/4$  for this particular point.

Let us assume now for simplicity that  $\lambda$  is a decreasing function of  $\mu$  only, and has a zero at  $\mu = \mu_*$ . If we consider the phase transitions for increasing  $T$  and fixed  $\mu$  we find a line of second-order phase transitions for  $\mu < \mu_*$ , and a line of first-order transitions for  $\mu > \mu_*$ . The two lines meet at the special point  $(T_*, \mu_*)$ , where  $T_* = T_{cr}$  for  $\mu = \mu_*$ . This point is characterized as a tricritical point. In the general case  $\lambda$  will be a decreasing function of a linear combination of  $\mu$  and  $T$ . The structure of the phase diagram remains the same. Simply there is a linear combination of  $T$  and  $\mu$  that generates displacements along the lines of first- and second-order phase transitions near the tricritical point, and a different one that moves the system through the phase transition.

If a source term  $-j\sigma$  is added to the potential of eq. (3) the  $O(4)$  symmetry is explicitly broken. The phase diagram is modified significantly even for small  $j$ . The second-order phase transitions, observed for  $\lambda > 0$ , disappear. The reason is that the minimum of the potential is always at a value  $\sigma \neq 0$  that moves close to zero for increasing  $T$ . For small


 Figure 1. *The phase diagram.*

$j$  the mass term at the minimum approaches zero at a value of  $T$  near what we defined as  $T_{cr}$  for  $j = 0$ . However, no genuine phase transition appears. Instead we observe an analytical crossover.

The line of first-order transitions persists for  $j \neq 0$ . However, at the critical temperature the minimum jumps discontinuously between two non-zero values of  $\sigma$  and never becomes zero. The line ends at a new special point, whose nature can be examined by considering the  $\sigma$ -derivative of the potential  $V(\rho)$  of eq. (3). For a first-order phase transition to occur,  $\partial V/\partial\sigma$  must become equal to  $j$  for three non-zero values of  $\sigma$ . This requires  $\partial^2 V/\partial\sigma^2 = 0$  at two values of  $\sigma$ . The critical point corresponds to the situation that all these values merge to one point. At the minimum  $\sigma_*$  of the potential  $\tilde{V}(\sigma; j) = V(\sigma, \pi_i = 0) - j\sigma$  at the critical point, we have:  $d\tilde{V}/d\sigma = d^2\tilde{V}/d\sigma^2 = d^3\tilde{V}/d\sigma^3 = 0$ . It can be checked that this requires  $\lambda < 0$  and can be achieved by fine-tuning  $m^2$  and  $\lambda$  for given  $j$ . The minimum  $\sigma_*$  is then completely determined. At the critical point we expect a second-order phase transition for the deviation of  $\sigma$  from  $\sigma_*$ . As  $d^4\tilde{V}/d\sigma^4 \neq 0$  at the critical point, the critical exponent  $\beta$  takes the value  $\beta = 1/2$ . The  $\pi$ 's are massive, because  $m_\pi^2 = \partial^2 V(\rho)/\partial\pi_i^2 = dV(\rho)/d\rho = j/\sigma_*$  at the critical point.

The phase diagram we discussed is presented in fig. 1. If we take into account the effect of fluctuations of the fields, the effective potential will have a more complicated form than

the simple classical potential of eq. (3). However, the qualitative structure of the phase diagram is not modified. The main modification is that the nature of the fixed points differs from the predictions of mean-field theory.

#### 4. THE RENORMALIZATION GROUP AND THE NATURE OF THE FIXED POINTS

The most efficient way to study a physical system near a phase transition is through the effective potential. In the formulation of the renormalization group which we are employing, the dependence of the potential of a three-dimensional theory on a “coarse-graining” scale  $k$  is described by the equation [9–11]

$$\frac{\partial}{\partial t} u_k(\tilde{\sigma}) = -3u_k + \frac{1}{2}(1 + \eta)\tilde{\sigma} u'_k + \frac{1}{4\pi^2} \left[ l_0^3(u''_k) + 3l_0^3(u'_k/\tilde{\sigma}) \right], \quad (6)$$

where  $t = \ln(k/\Lambda)$  and we have defined the dimensionless quantities

$$u_k = k^{-3}U_k, \quad \tilde{\sigma} = k^{-\frac{1}{2}}Z_k^{\frac{1}{2}}\sigma. \quad (7)$$

Primes denote derivatives with respect to  $\tilde{\sigma}$ . The scale-dependent potential  $U_k$  results from the integration of the field fluctuations with characteristic momenta larger than  $k$ . At the ultraviolet scale  $\Lambda$  it is identified with the classical potential:  $U_\Lambda = V$ . In the limit  $k \rightarrow 0$  it becomes equal to the effective potential:  $U_{eff} = U_0 \equiv U$ .

The three-dimensional theory described by the potential  $U_k$  results from the four-dimensional theory at non-zero temperature  $T$  at energy scales below  $T$  (dimensional reduction). The ultraviolet scale  $\Lambda$  can be taken equal to the temperature  $T$ . The initial condition  $U_\Lambda$  that is needed for the solution of the evolution equation (6) is given by eq. (3) [7,8].

The “threshold” function  $l_0^3(w)$  in eq. (6) falls off for large values of  $w$  following a power law. As a result it introduces a threshold behaviour for the contributions of massive modes to the evolution equation. In eq. (6) we distinguish contributions from two different types of fields: the radial mode (the  $\sigma$ -field) and the three Goldstone modes (the pions). Their masses are expressed through the derivatives of the effective potential.

The phase diagram derived through the renormalization-group study has all the expected features. For zero current quark masses there is a line of second-order phase transitions starting on the  $\mu = 0$  axis. It can be confirmed that they belong to the  $O(4)$  universality class by calculating universal quantities such as the critical exponents  $\beta$  and  $\nu$ . For large  $\mu$  there is a line of first-order phase transitions.

The two lines meet at a tricritical point, with specific values  $(\mu_*, T_*)$ , where a second-order phase transition takes place, governed by the Gaussian fixed point. This results in mean-field behaviour, as can be checked by calculating the relevant critical exponents. For  $\mu$  slightly smaller than  $\mu_*$  one observes universal crossover behaviour (not to be confused with the analytical crossover), as the initial influence of the Gaussian fixed point is slowly dominated by the more stable  $O(4)$  fixed point near the critical temperature. For  $\mu$  slightly larger than  $\mu_*$  very weak first-order phase transitions take place, for which the discontinuity in the order parameter approaches zero.

When the effect of non-zero current quark masses is taken into account by introducing a source term  $-j\sigma$ , the second-order phase transitions turn into an analytical crossover,

while the first-order ones retain their qualitative character. One can study in detail the emergence of the critical point at the end of the line of first-order phase transitions. The pions are massive near the critical point and they decouple from the evolution equation. A second-order phase transition takes place at the critical point, which belongs to the Ising universality class.

## 5. THE UNIVERSAL EQUATION OF STATE AND IMPLICATIONS FOR HEAVY-ION COLLISIONS

All the physical information near a second-order phase transition can be encoded in the equation of state. This describes the relation between the order parameter (in our case the field expectation value), the parameters that control the distance from the phase transition and the external source. In the case of one relevant parameter (which we denote by  $\delta m^2$ ) the equation of state can be cast in the form [12]

$$\delta j = \frac{dU}{d\sigma} = \sigma |\sigma|^{\delta-1} f(x), \quad x = \frac{\delta m^2}{|\sigma|^{1/\beta}}, \quad (8)$$

with  $U(\sigma)$  the effective potential and  $\delta j$  the external source. The function  $f(x)$  has a universal form determined by the universality class. The critical exponents  $\delta$ ,  $\beta$ , as well as the other critical exponents and amplitudes encoded in  $f(x)$ , also take values characteristic of the universality class.

The critical point of QCD corresponds to a certain non-zero value of the source that is proportional to the current quark mass. The pions remain massive even at the critical point and decouple at low energy scales. This leads to a critical theory with only one massless field, the  $\sigma$ -field. According to our discussion in section 3, for a non-zero current quark mass the chiral symmetry is never restored. This means that the critical point corresponds to a non-zero expectation value of the  $\sigma$ -field. As a result, the universal equation of state is more conveniently parametrized as

$$\delta j = \frac{d\tilde{U}}{d\sigma} = F_2(\epsilon, \delta m^2) + (\sigma + \epsilon) |\sigma + \epsilon|^{\delta-1} f_{Z_2}(x), \quad x = \frac{\delta m^2}{|\sigma + \epsilon|^{1/\beta}}, \quad (9)$$

with  $\delta m^2$ ,  $\epsilon$  proportional to deviations of the temperature  $\delta T$  and the baryonic chemical potential  $\delta \mu$  from certain values. The source term  $\delta j$  is proportional to the deviation of the quark mass from a constant value. This result is valid for  $\delta m^2, \epsilon \rightarrow 0$  and  $|\sigma|$  taking values in a range of a few  $|\epsilon|$ . The universal function  $f_{Z_2}(x)$  is specified by the Ising universality class [11,13]. The critical exponents are  $\beta = 0.33$ ,  $\delta = 4.8$ . The function  $F_2(\epsilon, \delta m^2)$  is not universal, and in principle could be incorporated in the parameter  $\delta j$ . We chose this parametrization, as  $\delta j$  is assumed to depend only on the current quark mass, and is not a parameter that can be varied in an experiment. We normalize  $F_2$  so that  $F_2(0, 0) = 0$ .

In experimental situations the critical point of QCD can be approached only along the surface  $\delta j = 0$ . The details of the experiment (center of mass energy, type of colliding nuclei) determine the effective temperature and chemical potential. Information from various experiments can be used in order to approach the critical point along a curve  $\epsilon = \epsilon(\delta m^2)$ . After a heavy-ion collision the central region, in which the phase transition is

expected to take place, expands and cools down. As a result, the system within this region is expected to follow a line in the  $(T, \mu)$  plane on which  $T$  and  $\mu$  are continuously reduced. This corresponds to a line in the  $(\epsilon, \delta m^2)$  plane. An important question is whether the system can pass sufficiently close to the critical point so that universal properties (such as scaling parametrized by critical exponents) are observable. The solution of eq. (9) with  $\delta j = 0$ ,  $\epsilon = \epsilon(\delta m^2)$  determines the location of the vacuum. The function  $F_2(\epsilon(\delta m^2), \delta m^2)$  is a regular function around the point  $(0,0)$  and can be Taylor expanded. As  $F_2(0,0) = 0$  the leading term is  $c \delta m^2$ , with  $c$  some constant. (We have assumed  $\epsilon = \epsilon(\delta m^2)$ .) The solution of eq. (9) is  $|\sigma + \epsilon(\delta m^2)| = |c \delta m^2 / D|^{1/\delta}$ , where  $D = f(0)$ . This solution emerges because the critical exponents  $\delta = 4.8$ ,  $\beta = 0.33$  satisfy  $\beta\delta > 1$ .

The “unrenormalized” mass term  $d^2\tilde{U}/d\sigma^2$  (equal to the inverse susceptibility) scales as  $|\sigma + \epsilon(\delta m^2)|^{\delta-1} \sim |\delta m^2|^{(\delta-1)/\delta}$ . This implies that the effective exponent  $\gamma_{eff}$  takes the value  $\gamma_{eff} = (\delta - 1)/\delta = 0.79$ . This should be compared with the standard value  $\gamma = 1.24$  in the Ising universality class [11,13]. For the exponent  $\nu$  parametrizing the divergence of the correlation length we find  $\nu_{eff} = 0.40$ . This is a consequence of the scaling law  $\nu = \gamma/(2 - \eta)$  that relates  $\nu$ ,  $\gamma$  and the small anomalous dimension  $\eta = 0.036$ . The standard value of  $\nu$  in the Ising universality class is  $\nu = 0.63$ .

It must be pointed out that the mass term may scale with the standard value for the exponent along certain paths that approach the critical point. For example, for  $\epsilon = 0$ ,  $\delta j = F_2(0, \delta m^2)$  the standard scaling is obtained. However, this approach to the critical point is unphysical as the quark mass cannot be altered. Another possible path has  $\delta j = 0$  and  $\epsilon(\delta m^2)$  given by the solution of the equation  $F(\epsilon, \delta m^2) = 0$ . Again, such a fine-tuned path is unlikely to be realized experimentally. These conclusions are in agreement with previous studies of the tricritical and critical points, in which the arguments were based on scaling relations [14].

The effective values for the exponents that we derived above are smaller than the standard ones in the Ising universality class by approximately 40%. This implies that the divergence of quantities such as the correlation length or the susceptibility along the experimentally accessible paths is much slower than the naive expectation. This conclusion is a consequence of the fact that the external source (the explicit symmetry breaking term) cannot be altered, as it is proportional to the quark mass. The smallness of the effective exponents implies that the universal behaviour is not easily accessible. The critical point must be approached very closely before the divergence of the correlation length or the susceptibility becomes apparent.

## REFERENCES

1. K. Rajagopal and F. Wilczek, arXiv:hep-ph/0011333.
2. D. U. Jungnickel and C. Wetterich, Phys. Rev. D **53** (1996) 5142 [arXiv:hep-ph/9505267]; Eur. Phys. J. C **1** (1998) 669 [arXiv:hep-ph/9606483]; Phys. Lett. B **389** (1996) 600 [arXiv:hep-ph/9607411].
3. J. Berges, D. U. Jungnickel and C. Wetterich, Eur. Phys. J. C **13** (2000) 323 [arXiv:hep-ph/9811347].
4. J. Berges, D. U. Jungnickel and C. Wetterich, Phys. Rev. D **59** (1999) 034010 [arXiv:hep-ph/9705474].



5. U. Ellwanger and C. Wetterich, Nucl. Phys. B **423** (1994) 137 [arXiv:hep-ph/9402221].
6. D. U. Jungnickel and C. Wetterich, arXiv:hep-ph/9610336.
7. N. Tetradis, Nucl. Phys. A **726** (2003) 93 [arXiv:hep-th/0303244].
8. N. Brouzakis and N. Tetradis, Nucl. Phys. A **742** (2004) 144 [arXiv:hep-th/0401074].
9. C. Wetterich, Phys. Lett. B **301** (1993) 90.
10. N. Tetradis and C. Wetterich, Nucl. Phys. B **422** (1994) 541 [arXiv:hep-ph/9308214].
11. J. Berges, N. Tetradis and C. Wetterich, Phys. Rept. **363** (2002) 223 [arXiv:hep-ph/0005122].
12. B. Widom, J. Chem. Phys. **43** (1965) 3898.
13. A. Pelissetto and E. Vicari, Phys. Rept. **368** (2002) 549 [arXiv:cond-mat/0012164];
14. M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. Lett. **81** (1998) 4816 [arXiv:hep-ph/9806219];