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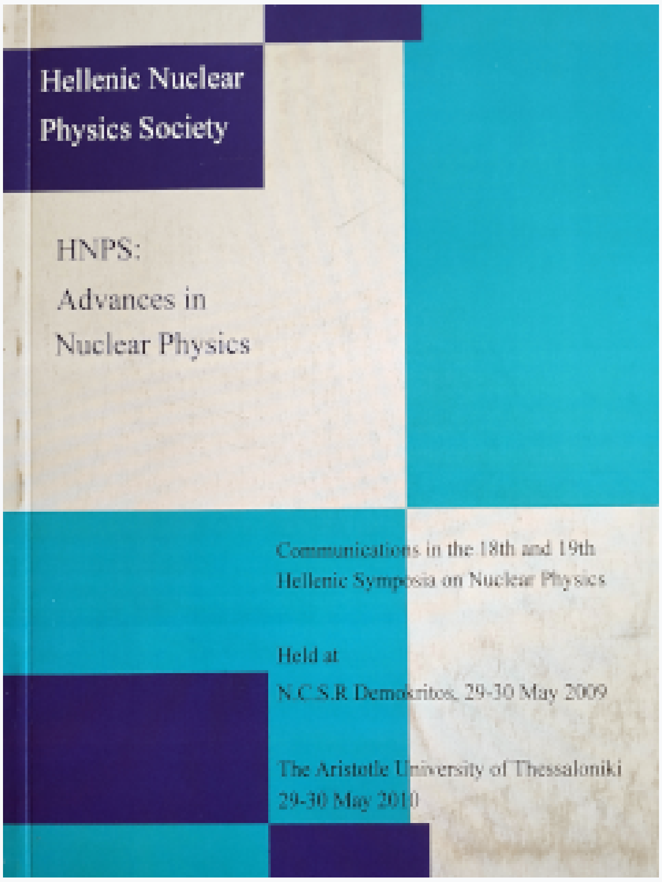


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# SUSYQM in nuclear structure: Bohr Hamiltonian with mass depending on the deformation

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**Abstract**

A well known problem of the Bohr Hamiltonian for the description of nuclear collective motion is that the nuclear moment of inertia increases with deformation too fast. We show that this can be avoided by allowing the nuclear mass to depend on the deformation. The resulting Hamiltonian is solved exactly, using techniques of Supersymmetric Quantum Mechanics.

## 1 Introduction

Deformation is not only a structural feature defined by the orientation of an organized nuclear system as a whole. It is also a reflection of mass differences between neighboring, almost spherical droplets of nuclear matter.

The liquid drop is pictured in the Bohr Hamiltonian [1,2]. An antithesis of this picture views the orientation of the nuclear system that causes its rotational spectra and the behavior of its moments of inertia. They are predicted to increase proportionally to $\beta^2$ [3], where $\beta$ is the collective variable corresponding to the axial deformation, while experimentally a much more moderate increase is observed.

What follows is the presentation of an effort to confront this antithesis. Motivation is based mainly in the classical limit of Interacting Boson Model (IBM) [4,5], where the momentum gets dressed with terms of deformation and thus, reveals the entrance of deformed algebras. It is also based on the argumentation developed in [6], for the experiment-based path to a non-constant mass coefficient in the Bohr Hamiltonian. The resulting method is dictated by the ”Quesne and Tkachuk equivalence” [7], which manifests that under certain circumstances, deformed canonical commutation relations are equivalent to a position dependent mass and also to a curved space.

## 2 The Quesne and Tkachuk equivalence and Position Dependent Mass Bohr Hamiltonian

A mathematicaly self-consistent way to obtain deformed operators is that of the deformed algebras, where the usual canonical quantization is modified by the presence of a so called deformation function. Quesne and Tkachuk [7] showed that if this deformation is a function of the position, that is

$$[\mathbf{x}, \mathbf{p}] = i\hbar f(\mathbf{x}), \tag{1}$$

then the following equivalence holds (the Quesne and Tkachuk equivalence)

$$f^2(\mathbf{x}) = \frac{1}{M(\mathbf{x})} = \frac{1}{g(\mathbf{x})}, \tag{2}$$

where $1/M(\mathbf{x})$ is an inverted position dependent mass and $1/g(\mathbf{x})$ is an inverted diagonalized metric. The consequence of the first part of this equivalence is that a deformed momentum operator of the type $\sqrt{f(\mathbf{x})}\mathbf{p}\sqrt{f(\mathbf{x})}$, is equivalent with the consideration of an effective mass $M(\mathbf{x}) = \frac{1}{(f(\mathbf{x}))^2}$. Following [7] the resulting Hamiltonian is of the form

$$H = -\frac{\hbar^2}{2m_0}\sqrt{f(\mathbf{x})}\nabla f(\mathbf{x})\nabla\sqrt{f(\mathbf{x})} + V_{eff}(\mathbf{x}), \tag{3}$$

with

$$\begin{aligned} V_{eff}(\mathbf{x}) = V(\mathbf{x}) + \frac{\hbar^2}{2m_0}\Big[&\frac{1}{2}(1-\delta-\lambda)f(\mathbf{x})\nabla^2 f(\mathbf{x}) \\ &+\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(\nabla f(\mathbf{x}))^2\Big]. \end{aligned} \tag{4}$$

The parameters $\delta$,$\lambda$ (von Roos parameters) manifest the mass dependence of the position and are analyzed in [7]. The application of the Position Dependent Mass (PDM) framework in the case of the Bohr Hamiltonian is presented in [8], for a mass dependent on the $\beta$ variable. First, a mass dependence of the form

$$B(\beta) = \frac{B_0}{(f(\beta))^2}, \tag{5}$$

where $B_0$ is a constant, is assumed. As the dependence is a scalar one, it permits us to follow the usual Pauli–Podolsky prescription. The PDM Bohr Hamiltonian is

$$\Bigg[-\frac{1}{2}\frac{\sqrt{f}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 f\frac{\partial}{\partial\beta}\sqrt{f} - \frac{f^2}{2\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{f^2}{8\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)} + v_{eff}\Bigg]\Psi = \epsilon\Psi, \quad (6)$$

where reduced energies $\epsilon = B_0 E/\hbar^2$ and reduced potentials $v = B_0 V/\hbar^2$ have been used, and

$$v_{eff} = v(\beta,\gamma) + \frac{1}{4}(1-\delta-\lambda)f\nabla^2 f + \frac{1}{2}\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(\nabla f)^2. \quad (7)$$

## 3 The deformed radial equation for the $\gamma$-unstable Davidson potential

The solution of the above Bohr-like equation can be reached for certain classes of potentials using techniques developed in the context of supersymmetric quantum mechanics (SUSYQM) [9, 10]. In order to achieve separation of variables, we assumed that the potential $v(\beta,\gamma)$ depends only on the variable $\beta$, i.e. $v(\beta,\gamma) = u(\beta)$ [11]. Such a choice is appropriate for the $\gamma$-unstable case. For the radial potential we used the Davidson potential [12]

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}, \quad (8)$$

which belongs to such a class. The radial equation takes its deformed version

$$HR = -\frac{1}{2}\left(\sqrt{f}\frac{d}{d\beta}\sqrt{f}\right)^2 R + u_{eff}R = \epsilon R, \quad (9)$$

with

$$u_{eff} = v_{eff} + \frac{f^2 + \beta f f'}{\beta^2} + \frac{f^2}{2\beta^2}\Lambda, \quad (10)$$

where $\Lambda = \tau(\tau+3)$, with $\tau$ being the seniority quantum number.

## 4 SUSY QM, Shape Invariance and the deformation function

Following the general method used in SUSYQM [9], one should first factorize the Hamiltonian in terms of generalized ladder operators. In order to obtain exact solutions, someone extents this factorization to a whole hierarchy of Hamiltonians with an integrability condition, the shape invariance. Shape invariance states that all the members of the hierarchy must retain the same functional dependence of the potential.

The deformation function is unknown. Its definition in Eq. (10) must respect the shape invariance condition, *i.e.*, bring the Davidson behavior to the $u_{eff}$. The choice

$$f(\beta) = 1 + a\beta^2, \tag{11}$$

gives the Davidson behavior to the effective potential,

$$\begin{aligned} u_{eff} = \beta^2 + a^2\beta^2 \left[\frac{5}{2}(1-\delta-\lambda) + 2\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right) + 3 + \frac{\Lambda}{2}\right] \\ + \frac{1}{\beta^2}\left(1 + \frac{1}{2}\Lambda + \beta_0^4\right) + a\left[\frac{5}{2}(1-\delta-\lambda) + 4 + \Lambda\right]. \end{aligned} \tag{12}$$

The parameter $a$ is called the deformation parameter. The mass is position dependent for non-zero values of $a$. Therefore the values of $\beta_0$ and $a$ show first the magnitude of the deformation, and second the position dependence of the mass. For tiny changes of $\beta_0$, the parameter $a$ varies in the Davidson behavior and this variation reflects the position dependence of the mass during the transition, as well as the deformation as an effect of mass differences. In Sec. 5 this situation is clarified. From Eq. (6) it is clear that in the present case the moments of inertia are not proportional to $\beta^2 \sin^2(\gamma - 2\pi k/3)$, but to $(\beta^2/f^2(\beta)) \sin^2(\gamma - 2\pi k/3)$. The function $\beta^2/f^2(\beta)$ is shown in Fig. 1 for different values of the parameter $a$. It is clear that the increase of the moment of inertia is slowed down by the function $f(\beta)$, as it is expected as nuclear deformation sets in [3].

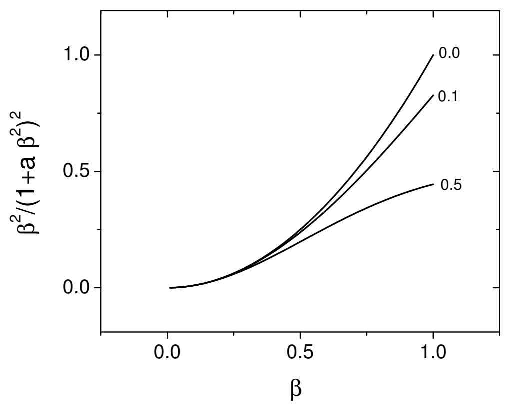


Fig. 1. The function $\beta^2/f^2(\beta) = \beta^2/(1+a\beta^2)^2$, to which moments of inertia are proportional as seen from Eq. (6), plotted as a function of the nuclear deformation $\beta$ for different values of the parameter $a$. See Sec. 4 for further discussion.

It is obvious now that the superpotential of the effective potential should be the one corresponding to the Davidson potential, which is known [10]. Therefore the ladder operators of the first member of the hierarchy should take the form [8]

$$B_0^{\pm} = \mp\frac{1}{\sqrt{2}}\left(\sqrt{f}\frac{d}{d\beta}\sqrt{f}\right) + \frac{1}{\sqrt{2}}\left(c_0\beta + \bar{c}_0\frac{1}{\beta}\right), \tag{13}$$

From these equations the eigenvalues and eigenfunctions are obtained in [8].

## 5 Numerical results

As a first testground of the present method we have used the Xe isotopes shown in Table 1. Their choice is justified in [8]. It is worth considering here the values of the parameters in each nucleus.

i) $^{134}$Xe and $^{132}$Xe are almost pure vibrators. Therefore no need for deformation dependence of the mass exists, the least square fitting leading to $a = 0$. Furthermore, no $\beta_0$ term is needed in the potential, the fitting therefore leading to $\beta_0 = 0$, *i.e.*, to pure harmonic behaviour.

ii) In the next two isotopes ($^{130}$Xe and $^{128}$Xe) the need to depart from the pure harmonic oscillator becomes clear, the fitting leading therefore to nonzero $\beta_0$ values. However, there is still no need of dependence of the mass on the deformation, the

Table 1
Comparison of theoretical predictions of the $\gamma$-unstable Bohr Hamiltonian with $\beta$-dependent mass (with $\delta = \lambda = 0$) to experimental data [13] of Xe isotopes. The $R_{4/2} = E(4_1^+)/E(2_1^+)$ ratios (indicated as $4/2$ in the table), as well as the quasi-$\beta_1$ and quasi-$\gamma_1$ bandheads, normalized to the $2_1^+$ state and labelled by $R_{0/2} = E(0_\beta^+)/E(2_1^+)$ and $R_{2/2} = E(2_\gamma^+)/E(2_1^+)$ respectively (indicated as $0/2$ and $2/2$ in the table), are shown. $n$ indicates the total number of levels involved in the fit and $\sigma$ is the rms quality measure.

| | $4/2$ exp | $4/2$ th | $0/2$ exp | $0/2$ th | $2/2$ exp | $2/2$ th | $\beta_0$ | $a$ | $n$ | $\sigma$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $^{118}$Xe | 2.40 | 2.32 | 2.5 | 2.6 | 2.8 | 2.3 | 1.27 | 0.103 | 19 | 0.319 |
| $^{120}$Xe | 2.47 | 2.36 | 2.8 | 3.4 | 2.7 | 2.4 | 1.51 | 0.063 | 23 | 0.524 |
| $^{122}$Xe | 2.50 | 2.40 | 3.5 | 3.3 | 2.5 | 2.4 | 1.57 | 0.096 | 16 | 0.638 |
| $^{124}$Xe | 2.48 | 2.36 | 3.6 | 3.5 | 2.4 | 2.4 | 1.55 | 0.051 | 21 | 0.554 |
| $^{126}$Xe | 2.42 | 2.33 | 3.4 | 3.1 | 2.3 | 2.3 | 1.42 | 0.064 | 16 | 0.584 |
| $^{128}$Xe | 2.33 | 2.27 | 3.6 | 3.5 | 2.2 | 2.3 | 1.42 | 0.000 | 12 | 0.431 |
| $^{130}$Xe | 2.25 | 2.21 | 3.3 | 3.1 | 2.1 | 2.2 | 1.27 | 0.000 | 11 | 0.347 |
| $^{132}$Xe | 2.16 | 2.00 | 2.8 | 2.0 | 1.9 | 2.0 | 0.00 | 0.000 | 7 | 0.467 |
| $^{134}$Xe | 2.04 | 2.00 | 1.9 | 2.0 | 1.9 | 2.0 | 0.00 | 0.000 | 7 | 0.685 |

fitting still leading to $a = 0$. Even if we have a finite value for the $\beta_0$, the mass is not yet position dependent. But in $^{126}$Xe, for the same value of $\beta_0$ as in $^{128}$Xe, the mass is position dependent. These three nuclei ($^{130}$Xe, $^{128}$Xe, and $^{126}$Xe) seem to be good candidates for the examination of the behavior of the mass during the phase transition from a spherical to a $\gamma$-unstable behavior.
iii) Beyond $^{126}$Xe, both the $\beta_0$ term in the potential and the deformation dependence of the mass become necessary, leading to nonzero values of both $\beta_0$ and $a$.

## 6 Conclusion

Based on the classical limit of the IBM and on the approach of a non constant mass coefficient in the Bohr Hamiltonian, a PDM Bohr Hamiltonian is obtained. Its application to $\gamma$-unstable nuclei gives encouranging results. Furthermore, the mass behaviour during the transition from the spherical to the $\gamma$-unstable case seems to be promising, indicating a dependence relationship between deformation and mass.

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