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# ANALYSIS OF THE BINDING ENERGIES OF THE $\Lambda$ - PARTICLE IN HYPERNUCLEI WITH THE RHVT APPROACH AND THE GAUSS POTENTIAL

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## Abstract

An analysis is carried out mainly of the ground state binding energies of the  $\Lambda$ -particle in hypernuclei with values of the core mass number  $A_c$  between 15 and 207 (included) using, as far as possible, recent experimental data. The renormalized (non- relativistic) quantum mechanical hypervirial theorem (RHVT) technique is employed in the form of  $s$ - power series expansions and a Gauss single particle potential for the motion of a  $\Lambda$ - particle in hypernuclei is used. Not exact analytic solution is known for the Schrödinger eigenvalue problem in this case. Thus, the approximate analytic expressions (AAE) for the energy eigenvalues which are obtained with the RHVT approach and are quite useful as long as the involved dimensionless parameter  $s$  is sufficiently small, are compared only with the numerical solution. The potential parameters are determined by a least-squares fit in the framework of the rigid core model for the hypernuclei. A discussion is also made regarding the determination of the renormalization parameter  $\chi$ .

## 1.Introduction

The Quantum mechanical hypervirial theorem (HVT) approach has been a very useful technique in dealing with various problems in Physics, Chemistry etc in which Quantum Mechanics provides the basic theoretical background. For pertinent reviews see refs [1,2].

This approach started originally with the work of Hirschfelder [3] and further elaborations and various applications followed (see e.g. ref.[4-8]). We mention in particular the use of this technique for the approximate treatment of problems such as the non relativistic motion of a particle in a potential  $V$ .

In more recent work [9-11] an investigation was carried out of the HVT approach in the form of approximate analytic expressions (AAE), originated from truncated expansions, when a small (compared to unity) dimensionless quantity  $s$  exists. Use was also made [12-13] of a renormalized HVT expansion (RHVT) for the energy eigenvalues of a particle moving in a single particle potential belonging to a fairly wide class of central potential wells, namely those of the form:

$$V(r) = -D f(r/R) \quad (1)$$

where  $D > 0$  is the potential depth,  $R > 0$  its "radius" and  $f$  ( $f(0)=1$ ) the potential form- factor which determines its shape and is assumed to be an appropriate analytic function of even powers of  $x = r/R$  with  $-d^2 f/dx^2|_{x=0} > 0$ , that is they behave like an harmonic oscillator near the origin.

Typical examples are the Gaussian and the (reduced) Poeschl-Teller (PT) potentials:

$$V(r) = -D e^{-(r/R)^2} \quad (a) \text{ and } V(r) = -D \cosh^{-2}(r/R) \quad (b) \quad (2)$$

In our first results we considered as an application to a physical problem the use of the derived AAE to estimate the binding energies of a  $\Lambda$ -particle in hypernuclei by assuming the PT single particle potential. The advantage in doing this, is that in certain cases (for the  $l=0$ , that is the  $s$ -states) there exist exact analytic expressions for this particle moving in this potential and a direct comparison of our AAE with them can be made to assess the accuracy of the latter. It should be clear, however, that the usefulness of the AAE refers mainly to the cases of the class of potentials (1) in which exact analytic results of the corresponding Schrödinger eigenvalues can not be found. It is the aim of the present work to use these AAE, mainly for the ground state  $\Lambda$ - particle moving in hypernuclei pertaining to the well-known potential (2a), taken as a first approximation of the mean -field felt by the  $\Lambda$  in hypernuclei.

In the following section a summary of the formalism is outlined. For more details one is referred to refs [12,13]. In section 3 the detailed preliminary numerical results are displayed for the  $\Lambda$  energies in a number of

hypernuclei mainly for the ground state along with the corresponding experimental results used. The final section is devoted to final remarks and a comment.

## 2. Outline of the formalism

We consider first the more general class of even – power series central potentials:

$$V(r) = -D + \sum_{k=0}^{\infty} V_k \lambda^k r^{2k+2} \quad (3)$$

This potential takes the following form, by adding and subtracting the same harmonic oscillator potential  $Kr^2$  (in a way analogous to that of ref [14,15] for the perturbed one dimensional oscillator):

$$V^R(r) = \left( -D + V_0^R r^2 \right) + \left[ \left( \sum_{k=1}^{\infty} V_k^R \lambda^k r^{2k+2} \right) - K r^2 \right] \quad (4)$$

where:  $V_0^R = V_0 + K$ ,  $V_k^R = V_k$  for  $k=1, 2, 3, \dots$

One introduces then the shifted energy eigenvalues

$$E_{n\ell}^{R'} = E_{n\ell}^R + D \quad (5)$$

and considers the corresponding radial Schrödinger equation and the corresponding equation for the relevant zeroth- order energy eigenvalues and eigenfunctions, (denoted by superscript (0)), and writes the expansions

$$E_{n\ell}^{R'} = \sum_{k=0}^{\infty} E^{R(k)} \lambda^k \quad (6)$$

$$\langle r^N \rangle_{n\ell}^R = \sum_{p=0}^{\infty} C_N^{R(p)} \lambda^p \quad (7)$$

On the basis of these expansions and the use of Hellmann-Feynmann theorem we end up after a rather lengthy algebra with the following recurrence relations which can be used to obtain the interested quantities:

$$\begin{aligned} (N+2)V_0 C_{N+2}^{R(k)} &= \frac{\hbar^2}{2\mu} N \left[ \frac{(N+1)(N-1)}{4} - \ell(\ell+1) \right] C_{N-2}^{R(k)} + (N+1)E^{R(0)} C_N^{R(k)} + \\ &+ \sum_{q=1}^k \left\{ (N+1)E^{R(q)} C_N^{R(k-q)} - (N+q+2)V_q C_{N+2q+2}^{R(k-q)} \right\} \end{aligned} \quad (8)$$

where:

$$C_0^{R(k)} = \delta_{k0} \text{ and } E^{R(q)} = \frac{1}{q} \sum_{m=1}^q m V_m C_{2m+2}^{R(q-m)}, \quad \begin{cases} k=0, 1, 2, 3, \dots \\ q=1, 2, 3, 4, \dots \\ m=1, 2, 3, 4, \dots \end{cases} \quad (9)$$

We focus now our attention to the class of potentials (1) mentioned in the introduction.

In this case we use the expression of the small (compared to unity) dimensionless quantity  $s$ :

$$s = \left( \hbar^2 / 2\mu D R^2 \right)^{1/2} \quad (10)$$

We can write after a lengthy algebra the expression of the energy eigenvalues in the RHVT approach as a truncated expression:

$$\frac{E_{n\ell}^R}{D} = e_{n\ell}^{R(0)} + e_{n\ell}^{R(1)} s + e_{n\ell}^{R(2)} s^2 + e_{n\ell}^{R(3)} s^3 + O(s^4) \quad (11)$$

where

$$e_{n\ell}^{R(0)} = -1 \quad (12)$$

$$e_{n\ell}^{R(1)} = 2 |d_I|^{1/2} a_{n\ell} \chi^{1/2} \quad (13)$$

$$e_{n\ell}^{R(2)} = -\frac{d_2}{8|d_1|} \left[ 3 - 4\ell(\ell+1) + 12a_{n\ell}^2 \chi \right] \quad (14)$$

$$e_{n\ell}^{R(3)} = \frac{a_{n\ell} \chi^{1/2}}{8|d_1|^{5/2}} \left\{ \left( 20d_1d_3 - 17d_2^2 \right) a_{n\ell}^2 \chi + \left( 25d_1d_3 - \frac{67}{4}d_2^2 \right) - 3\ell(\ell+1) \left( 4d_1d_3 - 3d_2^2 \right) \right\} \quad (15)$$

and  $a_{n\ell} = (2n + \ell + 3/2)$ . The new renormalization parameter  $\chi$  (which should depend, in general, on quantum numbers  $n\ell$ , that is  $\chi = \chi_{n\ell}$ ) is related with the Killingbeck renormalization parameter  $K$  with the relation:

$$\chi = 1 + \frac{K}{\left( \frac{|d_1|D}{R^2} \right)} \quad (16)$$

and it is seen that this parameter enters formula (11) only as a factor of the state dependent number  $a_{n\ell}^2$ .

Finally, the numbers  $d_k$  are related to the derivatives of  $f$  and are defined as follows:

$$d_k = \frac{1}{(2k)!} \frac{d^{2k}}{dx^{2k}} f(x) \Big|_{x=0}, k=0,1,2,3,\dots \quad (17)$$

It is seen that when  $K=0$  that is when  $\chi=1$  formula (11) becomes identical to the one in the HVT case [9-11]. Thus, depending on the value of  $\chi$  formula (11) incorporates both the HVT and the RHVT results.

### 3. Results with the Gaussian single particle potential

In the case of the Gaussian potential, the numbers  $d_k$  given by expression (17) in the previous section reduce to the following:  $d_k = (-1)^k/k!$  see refs. [16, 11]. The first of them are given in Table 1.

**TABLE 1**

$d_k$	Gaussian
$d_0$	1
$d_1$	-1
$d_2$	1/2
$d_3$	-1/6

Thus, the energy expressions for the ground state of the  $\Lambda$  (the 1s state)  $E_{00}^R$  and for the first excited state, the 1p state,  $E_{01}^R$  and also the absolute value of their difference:  $\Delta_{ps} = |E_{01} - E_{00}|$  which enters the so-called Bertlmann and Martin inequalities (see e.g. [17 – 19]) are easily followed from our general truncated expression (11) and are quite simple AAE.

Our detailed numerical results are displayed in Table 2, Table 3 and Table 4. In these tables we use the ground state experimental energy data for the determination of the potential parameters both for the 1s and 1p states. The same assumption was also made in refs [22, 23]. It should be also noted that, for the mass  $\mu$ , the reduced mass of the  $\Lambda$ -core system was used.

We use, as far as possible, recent experimental data [20, 21] for the ground state energy of a  $\Lambda$  in a number of hypernuclei (displayed in Table 4), which are quite numerous, in order that we determine, by least squares fitting, the potential parameters  $D$  and  $r_0$  in the framework of the rigid core model:  $R = r_0 A_c^{1/3}$ . If we use the “almost exact” (that is apart from the usual numerical inaccuracies) numerical results (see Table 2) obtained with a pertinent subroutine, we find the following best fit values  $D = 38.717 \text{ MeV}$ ,  $r_0 = 1.01646 \text{ fm}$ . The corresponding value of  $\chi^2$  (which we shall denote by  $F_{00}$  in order not to confuse that with the square of the renormalization parameter) is  $F_{00} = 4.58342$ .

Regarding the RHVT results, these depend on the assumed value of the renormalization parameter  $\chi$ .

For purposes of a rough estimate we can follow the procedure discussed for the PT potential, taking into account that now the potential shape is different. In the case of Gauss potential an approximate value of  $\chi^G$  is therefore  $\chi^G = 1 - e^{-1} = 0.632121$ , which is a little larger than the PT one. The best-fit values of the potential parameters obtained using a least squares –fitting of the ground state energies of the lambda particle to the corresponding experimental results are the following:  $D = 38.981 \text{ MeV}$ ,  $r_0 = 0.79150 \text{ fm}$  and  $F_{00} = 4.53671$ .

**Table 2**  
Detailed numerical results for the 1s and 1p states with the Gauss potential.

Mass number of the core nucleus	Numerical solution of the Schroedinger eigenvalue problem for the 1s state. Best fit values : $D = 38.717 \text{ MeV}$ , $r_0 = 1.01646 \text{ fm}$ $F_{00} = 4.58342$		Numerical solution of the Schr. e.p. for the 1p state with the previous D and $r_0$ .	The lowest energy level spacing
$A_C$	$s$ (From Expr. (10))	$E_{00} \text{ (MeV)}$	$E_{01} \text{ (MeV)}$	$\Delta_{ps} =  E_{01} - E_{00} $ (MeV)
15	0.278202	-12.420	-0.4838	11.936
27	0.224941	-16.465	-4.835	11.630
31	0.214232	-17.339	-5.935	11.404
39	0.197697	-18.726	-7.757	10.969
50	0.181391	-20.139	-9.690	10.449
55	0.175533	-20.658	-10.420	10.238
88	0.149482	-23.032	-13.830	9.202
138	0.128353	-25.039	-16.820	8.219
207	0.111967	-26.645	-19.270	7.375

We recall also the remarks made for the determination of the renormalization parameter  $\chi$  in ref [13], where the results were obtained with the PT potential.

In the first columns of Table 3, the results are displayed, for purposes of comparison, with the first two leading terms in the HVT ( $\chi=1$ ) approximation for the ground and the first excited state of the lambda particle in the hypernuclei, as well as the results of  $\Delta_{ps}$ . It is easily seen that in this case, the formulae used are identical with those in the PT potential and the results with the Gauss potential are exactly the same as it is also immediately realized.

Regarding the RHVT case (displayed in the last columns of Table 3), we use the expression given in ref.[13]:

$$E_{n\ell}^R \cong D \left\{ -1 + 2 \left| d_1 \right|^{1/2} a_{n\ell} \chi_{n\ell}^{1/2} s \right\} \quad (18)$$

where now  $\chi_{n\ell}^{1/2} = \chi_{n\ell}^{1/2G}$  and we have:

$$\chi_{00}^{1/2G} \cong 1 - \frac{5}{8} s - \frac{25}{192} s^2 + O(s^3) \quad (19)$$

$$\chi_{01}^{1/2G} \cong 1 - \frac{7}{8} s - \frac{21}{64} s^2 + O(s^3). \quad (20)$$

This is equivalent to the HVT results in which the  $s^2$  and  $s^3$  terms are also included through the expressions of the renormalization parameters. The best-fit values of the potential parameters of the ground – state energy values (taken to be the same for the  $E_{01}$  as previously) are:  $D = 38.587 \text{ MeV}$ ,  $r_0 = 1.02392 \text{ fm}$  and  $F_{00} = 4.65636$ . It is seen that the AAE results of the RHVT case we are considering, are quite close to those obtained with the numerical subroutine for the determination of the energy eigenvalues of the ground state of the  $\Lambda$  in the hypernuclei. It is also seen that the behaviour of the  $\Delta_{ps}$  is as expected. Furthermore, the values of this quantity are a little larger than the “almost exact” numerical ones for  $A_c < 55$  and less than those for  $A_c > 55$ . However, the absolute values of the corresponding differences are, in most cases, more pronounced than those obtained with the PT potential.

**Table 3**  
Detailed numerical results based on simple approximate analytic expressions (AAE) and using the Gauss potential.

$A_C$	s	$E_{00}^{HVT}$ (MeV)	$E_{01}^{HVT}$ (MeV)	$\Delta_{ps}^{HVT}$ (MeV)	s	$E_{00}^{RHVT}$ (MeV)	$E_{01}^{RHVT}$ (MeV)	$\Delta_{ps}^{RHVT}$ (MeV)
15	0.21589	-12.399	2.796	15.195	0.27664	-12.419	0.527	12.946
27	0.17456	-16.672	-4.477	12.195	0.22368	-16.482	-4.587	11.895
31	0.16625	-17.640	-5.939	11.701	0.21303	-17.356	-5.759	11.597
39	0.15342	-18.994	-8.197	10.797	0.19657	-18.740	-7.664	11.076
50	0.14076	-20.330	-10.423	9.907	0.18037	-20.149	-9.651	10.498
55	0.13622	-20.810	-11.223	9.587	0.17455	-20.666	-10.391	10.275
88	0.11600	-22.944	-14.780	8.164	0.14864	-23.028	-13.845	9.183
138	0.09960	-24.675	-17.665	7.010	0.12763	-25.022	-16.844	8.178
207	0.08689	-26.018	-19.903	6.115	0.11134	-26.616	-19.286	7.330

**Table 4**  
The experimental results of the binding energies along with the theoretical values of the energies of the  $\Lambda$  with the Gauss Potential for the s and the p states and of the quantities  $e_{00}$ ,  $e_{01}$  (see table 5 of ref [13]) for various values  $A_C$

$A_C$	$B^{exp} \pm \Delta B^{exp}$ (MeV)	$E_{00}$ (MeV)	$E_{00}^{RHVT}$ (MeV)	$e_{00} = E_{00}^{''ex''} - E_{00}^{RHVT}$ (MeV)	$e_{00}(\%)$	$E_{01}$ (MeV)	$E_{01}^{RHVT}$ (MeV)	$e_{01} = E_{01}^{''ex''} - E_{01}^{RHVT}$ (MeV)	$e_{01}(\%)$
15	12.42±0.05	-12.420	-12.419	-0.001	0.008	-0.4838	0.527	-1.011	208.929
27	16.6±0.2	-16.465	-16.482	0.017	-0.103	-4.835	-4.587	-0.248	5.129
31	17.5±0.5	-17.339	-17.356	0.017	-0.098	-5.935	-5.759	-0.176	2.965
39	18.70±1.1	-18.726	-18.740	0.014	-0.075	-7.757	-7.664	-0.093	1.199
50	19.97±0.13	-20.139	-20.149	0.010	-0.050	-9.690	-9.651	-0.039	0.402
55	21.15±1.5	-20.658	-20.666	0.008	-0.039	-10.420	-10.391	-0.029	0.278
88	23.11±0.1	-23.032	-23.028	-0.004	0.017	-13.830	-13.845	0.015	0.108
138	23.8±1.0	-25.039	-25.022	-0.017	0.068	-16.820	-16.844	0.024	0.143
207	26.5±0.5	-26.645	-26.616	-0.029	0.109	-19.270	-19.286	0.016	0.083

We mention, finally, that the differences between the values of the RHVT ground state energies of the  $\Lambda$  and the corresponding ones obtained numerically are quite small as are with the PT potential, although with the Gauss potential these differences are more pronounced. The differences are bigger in the  $E_{01}$  energies, as we can see from Tables 3 and 4 of this work (see also Table 4 and Table 5 of ref [13] and also the relevant comments there).

#### 4. Final remarks and a comment

From the results with the Gauss potential displayed in the previous section it can be realized that the least - squares fit values  $F_{00}$  (both in the numerical and the RHVT cases) are a little smaller than the corresponding ones obtained with the PT potential (for which the numerical solution gives  $F_{00}=4,74393$  and the RHVT gives  $F_{00}=4,75062$ ) and in this sense the Gauss potential might be preferable in comparison with the PT one. It should be mentioned, however, that the RHVT results differ more from the “almost exact “numerical ones. We mention also that the AAE used come from an appropriate truncation of the power series expansions. Another possibility would be to consider an HVT energy expression which has its finite terms ( $s^3$  included) to coincide with the existing HVT ones plus an analytic expression, consisting only of higher  $s$  terms (which emerge by means e.g. of the binomial formula) which differ, however, from those of the original HVT expansion. We are currently investigating this possibility to see whether improvements can be obtained.

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