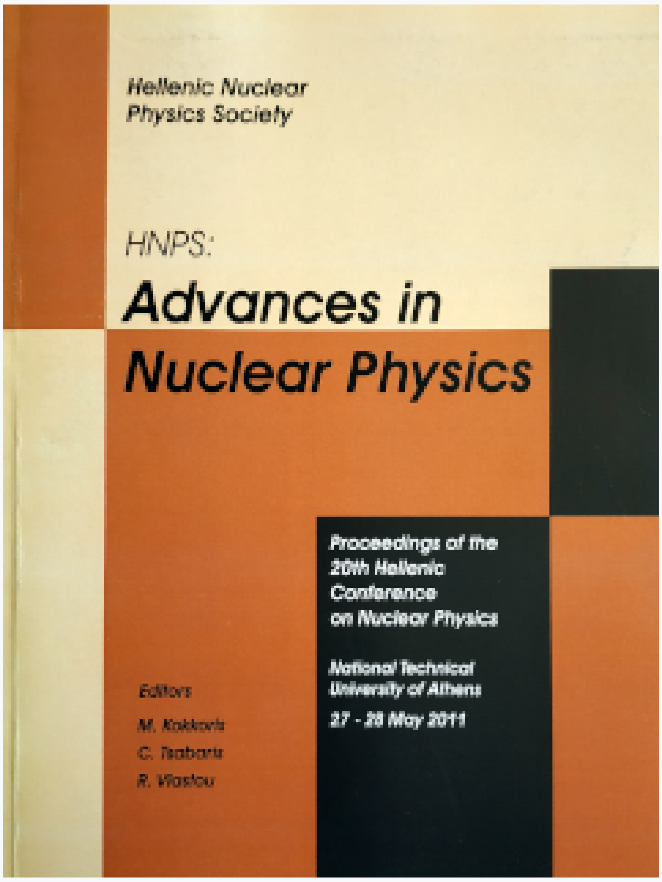
HNPS Advances in Nuclear Physics

Vol 19 (2011)

HNPS2011



Nilsson orbitals in quantum superintegrable systems

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doi: [10.12681/hnps.2512](https://doi.org/10.12681/hnps.2512)

Nilsson orbitals in quantum superintegrable systems

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Abstract

Recent studies [1] of the valence proton-neutron interaction through certain double differences of binding energies suggest that Nilsson orbitals with $\Delta K[\Delta N \Delta n_z \Delta\Lambda] = 0[110]$ are responsible for large overlaps of wave functions in heavy deformed nuclei. We show that pairs of orbitals of this kind belong to neighboring irreducible representations of appropriate deformed U(3) algebras [2], being interconnected by deformed creation and annihilation operators. These deformed U(3) algebras correspond to oscillators with rational ratios of frequencies, which are examples of quantum superintegrable systems [3].

Key words: Quantum superintegrable systems, Nilsson model, proton–neutron interaction.

1 Introduction to Quantum Superintegrable Systems

In the case of autonomic quantum systems ($H \neq H(t)$), if the commutator between the Hamiltonian and some physical quantity vanishes, $[H, I_k] = 0$, then the physical quantity I_k is a constant of motion due to Liouville's theorem. In general the maximum number of these constants is equal to the degrees of freedom of the system. Well known examples of Quantum Integrable Systems are one dimensional Spin Chain systems, Solitons in Yang Mills theories and many other cases.

In many cases there are systems which hold more integrals of motion than degrees of freedom. These systems are called **SUPERINTEGRABLE** and the extra integrals of motion are defined by C_m, where $m = 1, \ldots, q$. In general there are two classes of these systems. Firstly, if $q = N - 1$ then the system is called *Maximally Superintegrable*, and secondly if $q = 1$ then the system is called *Minimally Superintegrable* (or *Exactly Solvable*). In the special case of Quantum Maximally Superintegrable Systems (Q.M.S.S) the total number of constants, "living" in an N dimensional space, is $k + q = N + (N - 1) = 2N - 1$, where k are the integrals connected with the degrees of freedom of the system and q are the extra integrals. Well known examples of Q.M.S.S. are the hydrogen atom and the harmonic oscillator in N dimensions (with or without curvature).

Before moving forward it is very important to explain why superintegrable systems are so important in mathematical physics studies. Firstly because for these systems we can find eigenvalues and eigenvectors using **ONLY** algebraic methods. Moreover because using these systems we can form new bases to define new special functions of mathematical physics. Finally they help us to describe completely new kinds of symmetries of quantum systems (beyond the usual geometrical symmetries)

1.1 How do we study Q.M.S.S. ?

In order to study Q.M.S.S. we use the general scheme of **deformed oscillator algebras**. For a two dimensional Q.M.S.S. with constants H, I, C we have:

$$[H, I] = [H, C] = 0, \quad [I, C] \neq 0. \tag{1}$$

Hypothesis: Consider a two dimensional Q.M.S.S. for which we can construct the associative algebra defined by $\{1, H, A, A^\dagger\}$

$$\begin{aligned} N &= N(H, I, C), \\ N^\dagger &= N, \\ A &= A(H, I, C), \\ [N, A] &= -A, \\ A^\dagger A &= \Phi(H, N), \\ [A^\dagger A, AA^\dagger] &= 0, \end{aligned} \tag{2}$$

where $\Phi(E, x)$ is a real and positive function, defined for $x \geq 0$, and $\Phi(E, 0) = 0$. From the above equations we demand

$$[N, A^\dagger] = A^\dagger, \quad AA^\dagger = \Phi(H, N + 1).$$

Once we have defined the algebra, we also need a representation of this algebra. Due to the fact that the algebra is "oscillator" type, the representation must be a **Fock-like type representation**. We define a Fock - type space for every eigenvalue of the energy

$$H|E,n>=E|E,n>,$$

$$N|E,n>=n|E,n>,$$

$$A|E,n>=\sqrt{\Phi(H,N)}|E,n-1>,$$

$$A^{\dagger}|E,n>=\sqrt{\Phi(H,N+1)}|E,n+1>,$$

$$A|E,0>=0,$$

$$|E,n>=(\frac{1}{\sqrt{[n]!}})(A^{\dagger})^{n}|E,0>,$$

where $[0]!=1,[n]!=\Phi(E,n)[n-1]!$. In the case of discrete eigenvalues of energy, for every eigenvalue E we have degeneracy of order N_d+1. This is equal to demanding $\Phi(E,N_d+1)=0$ [2].

At this point we apply the previous ideas in two specific problems, the quantum anisotropic harmonic oscillator with rational ratios of frequencies with or without Rosochatius interaction terms. The Hamiltonian for this system is

$$H=\sum_{k=1}^{N}(\frac{p_k^2}{2}+\frac{1}{2}m\omega_0^2 n_k^2 x_k^2+\frac{\lambda_k}{x_k^2}), \tag{3}$$

where n_k $(k=1\ldots,N)$ are rational numbers, while λ_k are coupling constants. Firstly we prove that these systems are superintegrable and then we find eigenvalues of the energy and a representation of the algebra. The case of an anisotropic oscillator without Rosochatius terms is connected with the Nilsson model of nuclear structure.

2 Connection with the Nilsson model

The Nilsson Hamiltonian contains three terms

$$H_{Nilsson}=H_{osc}-2k\overrightarrow{L}\cdot\overrightarrow{S}-k\nu\overrightarrow{L}^2. \tag{4}$$

In the case of heavy nuclei the spin orbit interaction and the term proportional to the square of the angular momentum operator can be neglected. Moreover, heavy nuclei correspond to large deformations when described in the framework of the Nilsson model, so the isotropic oscillator is inappropriate for them. As a result, we must use the anisotropic oscillator. So there is a correspondence between the anisotropic oscillator with rational ratios of frequencies and the Nilsson model in the limit of heavy nuclei.

In the Nilsson model we use the Nilsson orbital notation $K[Nn_z\Lambda]$, where N is the principal quantum number of the major shell (or the number of oscillator quanta, $1\hbar\omega$, $2\hbar\omega$,...), n_z is the number of nodes in the wave function along the z axis, Λ is the projection of the orbital angular momentum l_i on the symmetry axis, l_{iz}, K is the projection of the total angular momentum j_i on the symmetry axis, j_{iz}.

An interesting quantity is the mass filter

$$|\delta V_{pn}(Z,N)| = \frac{1}{4}[(B_{Z,N} - B_{Z,N-2}) - (B_{Z-2,N} - B_{Z-2,N-2})].$$

This function is defined in such a way that double differences of masses are designed to isolate a particular interaction. For the proton–neutron interaction in even–even nuclei we use δV_{pn}.

It is well known that certain double differences of binding energies, known as $\delta V_{pn}(Z,N)$, show singularities in light even–even nuclei with $N = Z$, attributed to the large overlaps of the proton and neutron wave functions. Recent studies [1] have revealed a similar phenomenon in heavy nuclei, when the valence nucleons are considered. It has been proposed that this arises from the similarity of the sequences of orbitals in different major shells, providing large overlaps when the relevant proton and neutron shells are approximately equally filled. In particular, it has been found that $\delta V_{pn}(Z,N)$ is maximized for the pairs of orbitals

$$(p: 1/2[411], n: 1/2[521]), (p: 7/2[523], n: 7/2[633]), (p: 7/2[404], n: 7/2[514]),$$

where the usual notation $K[Nn_z\Lambda]$ is used. It has been remarked [1] that these pairs correspond to orbitals with $\Delta K[\Delta N\Delta n_z\Delta\Lambda] = 0[110]$.

2.1 The tools

In order to study this problem from the superintegrable point of view, we define the following Hamiltonian for the anisotropic oscillator

$$H = \frac{1}{2}\sum_{k=1}^{N}\left({p_k}^2 + \frac{x_k^2}{m_k^2}\right), \tag{5}$$

where m_k are natural numbers, any pair of them having no common divisor other than 1, and the ladder operators

$$a_k^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x_k}{m_k} - ip_k\right), \qquad a_k = \frac{1}{\sqrt{2}}\left(\frac{x_k}{m_k} + ip_k\right), \tag{6}$$

$$[a_k, a_k^\dagger] = \frac{1}{m_k}, \tag{7}$$

$$U_k = \frac{1}{2}\left({p_k}^2 + \frac{x_k^2}{m_k^2}\right) = \frac{1}{2}\left\{a_k, a_k^\dagger\right\}, \tag{8}$$

in terms of which the Hamiltonian can be written as

$$H = \sum_{k=1}^{N} U_k. \tag{9}$$

The 6 operators A_k, $A_k^\dagger$,

$$A_k^\dagger = \left(a_k^\dagger\right)^{m_k}(a_3)^{m_3}, \qquad A_k = (a_k)^{m_k}\left(a_3^\dagger\right)^{m_3}, \qquad k = 1, 2 \tag{10}$$

$$A_3^\dagger = (a_1^\dagger)^{m_1}(a_2)^{m_2}, \qquad A_3 = (a_1)^{m_1}(a_2^\dagger)^{m_2}, \tag{11}$$

together with the 3 operators U_k ($k = 1$, 2, 3), define a deformed $U(3)$ algebra [2].

We have considered in detail the cases (1:1:2), (1:1:3) and (2:2:1), for which the relevant algebras have been explicitly constructed. These algebras are not Lie algebras, since the commutators of generators result in general in powers of generators higher than 1 in the rhs, but their represention have been constructed and studied in detail.

Furthermore we define the operators

$$L_k = i\epsilon_{ijk}\left((a_i)^{m_i}\left(a_j^\dagger\right)^{m_j} - \left(a_i^\dagger\right)^{m_i}(a_j)^{m_j}\right), \tag{12}$$

$$[L_i, L_j] = i\epsilon_{ijk}\left(F(m_k, U_k + 1) - F(m_k, U_k)\right)L_k, \tag{13}$$

where the function $F(m, U)$ is defined by

$$F(m, U) = \prod_{p=1}^{m}\left(U - \frac{2p-1}{2m}\right). \tag{14}$$

In order to see if our procedure is close to physical reality we consider some special values for parameters m_k

- In the case of $m_1 = m_2 = m_3 = 1$ one has $F(1,x) = x - 1/2$, so that the above equation gives the usual angular momentum commutation relations

$$[L_i, L_j] = i\epsilon_{ijk} L_k. \tag{15}$$

- In the case $m_1 = m_2 = 1$ and $m_3 = m$, which is used to describe axially symmetric prolate nuclei, the quantity

$$L^2 = L_1^2 + L_2^2 + (F(m_3, U_3 + 1) - F(m_3, U_3))L_3^2 \tag{16}$$

  satisfies the commutation relations

$$[L^2, L_i] = 0, \qquad i = 1, 2, 3. \tag{17}$$

- The case $m_1 = m_2 = m$ and $m_3 = 1$ is used to describe axially symmetric oblate nuclei.

3 Solution, conclusions, open problems

Nilsson orbitals with $\Delta K[\Delta N \Delta n_z \Delta\Lambda] = 0[110]$, proposed to be responsible for large overlaps of wave functions in heavy deformed nuclei, are studied in the framework of oscillators with rational ratios of frequencies. These orbitals are shown to belong to neighboring irreducible representations of appropriate deformed U(3) algebras, being interconnected by deformed creation and annihilation operators. In the special case of 1:1:m oscillators, the L_3 operator of the deformed U(3) algebra is the usual physical angular momentum operator. Thus, in these special cases, the proof regards usual Nilsson orbitals. In a strange way studies in quantum superintegrable systems are connected with the p–n interaction problem (at least in the limit of heavy nuclei). Is it a coincidence or does it exist a deeper connection between Q.M.S.S. and nuclear structure?

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