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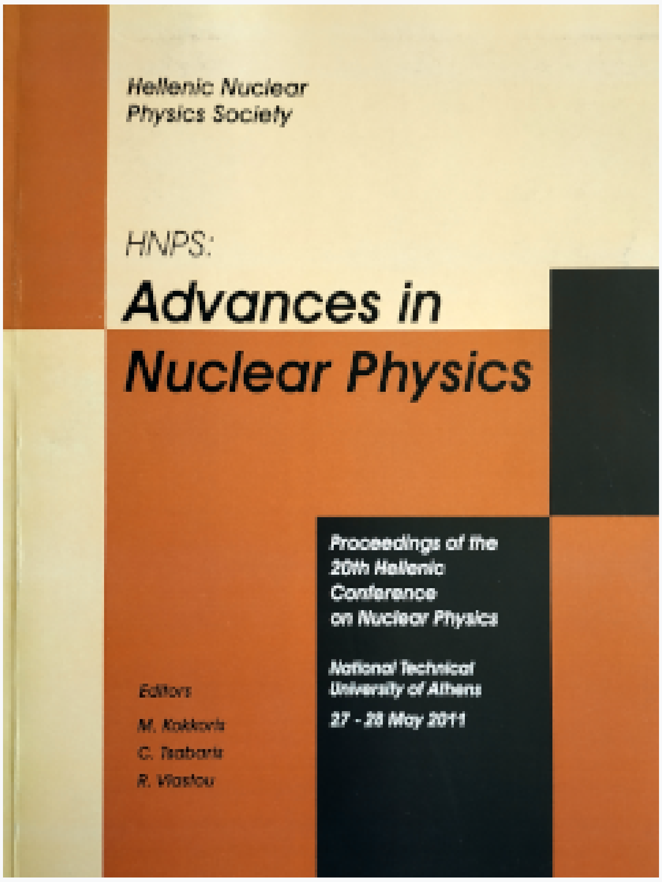

Group contractions and conformal maps in nuclear structure models

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Abstract

Group contraction is a procedure in which a symmetry group is reduced into a group of lower symmetry in a certain limiting case. Examples are provided in the large boson mumber limit of the Interacting Boson Approximation (IBA) model by a) the contraction of the SU(3) algebra into the [R^5]SO(3) algebra of the rigid rotator, consisting of the angular momentum operators forming SO(3), plus 5 mutually commuting quantities, the quadrupole operators, b) the contraction of the O(6) algebra into the [R^5]SO(5) algebra of the γ-unstable rotator. We show how contractions can be used for constructing symmetry lines in the interior of the symmetry triangle of the IBA model.

In mathematics, a conformal map is a function which preserves angles. We show how this procedure can be used in the framework of the Bohr Hamiltonian, leading to a Hamiltonian in a curved space, in which the mass depends on the nuclear deformation β, while it remains independent of the collective variable γ and the three Euler angles. This Hamiltonian is proved to be equivalent to that obtained using techniques of Supersymmetric Quantum Mechanics.

Key words: Group contactions, Interacting Boson Approximation model, conformal maps, Bohr Hamiltonian.

1 Group contractions

The SU(3)→ [R^5]SO(3) contraction [1,2] is a procedure in which the full SU(3) algebra, consisting of 8 noncommuting generators, is shrinked into an SO(3) algebra (consisting of 3 noncommuting generators), accompanied by 5 mutually commuting operators (the quadrupole operators). This simplification occurs in the limit of large boson number in which, in SU(3), all intrinsic excitations rise in energy, isolating the ground state band so that SU(3) goes over, approximately, into a simple rigid rotator. The resulting algebraic structure is, indeed, known [3] to be the algebra of the rigid rotator.

The SU(3) commutation relations read

$$[\hat{L}_\xi, \hat{L}_\nu] = -\sqrt{2}(1\xi 1\nu|1\xi+\nu)\hat{L}_{\xi+\nu}, \tag{1}$$

$$[\hat{L}_\xi, \hat{Q}^{(2)}_{SU(3),\nu}] = -\sqrt{6}(1\xi 2\nu|2\xi+\nu)\hat{Q}^{(2)}_{SU(3),\xi+\nu}, \tag{2}$$

$$[\hat{Q}^{(2)}_{SU(3),\xi}, \hat{Q}^{(2)}_{SU(3),\nu}] = \frac{3}{4}\sqrt{\frac{5}{2}}(2\xi 2\nu|1\xi+\nu)\hat{L}_{\xi+\nu}. \tag{3}$$

The second order Casimir operator is

$$\hat{C}_2[SU(3)] = \frac{2}{3}\left[2\hat{Q}^{(2)}_{SU(3)} \cdot \hat{Q}^{(2)}_{SU(3)} + \frac{3}{4}\hat{L}\cdot\hat{L}\right], \tag{4}$$

while its eigenvalues in the Elliott basis, (λ, μ), are

$$C_2(\lambda,\mu) = \frac{2}{3}(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu). \tag{5}$$

If we consider SU(3) irreducible representations (irreps) with large values of $C_2(\lambda,\mu)$, that is for large boson numbers, we can rescale the quadrupole operator as

$$\hat{q}^{(2)}_{SU(3),\xi} = \frac{\hat{Q}^{(2)}_{SU(3),\xi}}{\sqrt{C_2(\lambda,\mu)}}. \tag{6}$$

The first two commutation relations remain unchanged by the rescaling, while the last one becomes

$$[\hat{q}^{(2)}_{SU(3),\xi}, \hat{q}^{(2)}_{SU(3),\nu}] = \frac{3}{4}\sqrt{\frac{5}{2}}(2\xi 2\nu|1\xi+\nu)\frac{\hat{L}_{\xi+\nu}}{C_2(\lambda,\mu)}. \tag{7}$$

Then in the limit of large values of $C_2(\lambda,\mu)$ one gets

$$[\hat{q}^{(2)}_{SU(3),\xi}, \hat{q}^{(2)}_{SU(3),\nu}] = 0. \tag{8}$$

This result, which is obtained for large boson number, is called the contraction of SU(3) to [R^5]SO(3), where [R^5]SO(3) is the algebra of the rigid rotator [3], generated by the angular momentum operators of SO(3) and the five commuting operators $\hat{q}^{(2)}_{SU(3),\xi}$, $\xi = -2,-1,0,1,2$.

An immediate consequence of Eqs. (7) and (8) is that, in the contraction limit, terms proportional to the angular momentum $\hat{L}$ can be ignored. In the IBA framework, in which $\hat{L}$ is proportional to $(d^\dagger \tilde{d})^1$, this implies that $(d^\dagger \tilde{d})^1$ terms can be ignored.

In the limit of large values of $C_2(\lambda,\mu)$ and $\lambda \geq \mu$ the intrinsic quadrupole moments become [2]

$$q_0 = \frac{1}{2\sqrt{2}}(2\lambda+\mu+3), \qquad q_2 = \frac{1}{4}\sqrt{3(\mu-K)(\mu+K+2)}, \tag{9}$$

where K is the eigenvalue of the angular momentum projection on the body-fixed z-axis, for which $K \leq L$ is valid, as one can see from the algorithm of the SU(3)$\supset$SO(3) reduction. For states with $\lambda >> L$ (therefore also $\lambda >> K$) and $\lambda >> \mu$ one then obtains [1]

$$q_0 = \frac{\lambda}{\sqrt{2}}, \tag{10}$$

while q_2 becomes negligible. Since the ground state band belongs to the $(2N,0)$ irreducible representation (irrep) of SU(3), while other low-lying bands belong to irreps $(2N-4i-6j, 2i)$, $i=$ 0,1,2,..., $j=$ 0,1,2,... with relatively low i, j, the contraction occurs in the large N limit. Thus in the case of interest the intrinsic quadrupole moment becomes

$$q_0 = N\sqrt{2}. \tag{11}$$

An equivalent statement is that one can approximately replace the operator $\hat{Q}^{(2)}_{SU(3)}$ by the scalar $\lambda/\sqrt{2}$, as one can see from Eqs. (4) and (5), since the terms containing $\hat{L}$ and μ are negligible in this limit, having as a consequence that only the first term in the rhs of these equations survives. A formal justification for this replacement is given in Ref. [4], where matrix elements of the commutators of the relevant parts of the Hamiltonian with the quadrupole operator are properly considered, resulting in the appearance of the intrinsic quadrupole moment.

It should be noticed that the above results have been obtained in irreps with $\lambda >> L$, thus they regard the low lying part of the spectrum.

A similar procedure is followed in the contraction of O(6) to [R^5]SO(5) [5,6]. This is a procedure in which the full O(6) algebra, consisting of 15 noncommuting generators, is shrinked into an SO(5) algebra (consisting of 10 noncommuting generators), accompanied by 5 mutually commuting operators (the quadrupole operators). The resulting algebraic structure is known [6] to be the algebra of the γ-unstable rotator.

The first contraction has been used in Ref. [4] for determining a line of approximate SU(3) symmetry inside the symmetry triangle of the IBA, as described by S. Karampagia in this conference [7].

2 Conformal maps

In Ref. [8] it has been proved that the position-dependent effective mass formalism can be equivalently expressed in a curved space. We shall prove here that this connection is possible also in the case of the Bohr Hamiltonian.

Ordering the coordinates as

$$q_1 = \Phi, \quad q_2 = \Theta, \quad q_3 = \psi, \quad q_4 = \beta, \quad q_5 = \gamma, \tag{12}$$

the kinetic energy in the standard Bohr Hamiltonian [9] can be represented as

$$T = \frac{B}{2}\left(\frac{ds}{dt}\right)^2, \tag{13}$$

where

$$ds^2 = g_{ij} dq_i dq_j, \tag{14}$$

the symmetric matrix g_{ij} having the form

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & 0 & g_{33} & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 \\ 0 & 0 & 0 & 0 & g_{55} \end{pmatrix}, \tag{15}$$

with

$$g_{11} = \frac{J_1}{B}\sin^2\Theta\cos^2\psi + \frac{J_2}{B}\sin^2\Theta\sin^2\psi + \frac{J_3}{B}\cos^2\Theta, \tag{16}$$

$$g_{12} = \frac{1}{B}(J_2 - J_1)\sin\Theta\sin\psi\cos\psi, \qquad g_{13} = \frac{J_3}{B}\cos\Theta, \tag{17}$$

$$g_{22} = \frac{J_1}{B}\sin^2\psi + \frac{J_2}{B}\cos^2\psi, \qquad g_{33} = \frac{J_3}{B}, \qquad g_{44} = 1, \qquad g_{55} = \beta^2, \tag{18}$$

where the moments of inertia are

$$J_k = 4B\beta^2 \sin^2\left(\gamma - k\frac{2\pi}{3}\right). \tag{19}$$

The determinant of the matrix is

$$g = \frac{J_1 J_2 J_3}{B^3}\beta^2 \sin^2\Theta = 4\beta^8 \sin^2 3\gamma \sin^2\Theta. \tag{20}$$

The relevant volume element is then

$$dV = 2\beta^4 \sin 3\gamma \sin\Theta d\Phi d\Theta d\psi d\beta d\gamma. \tag{21}$$

The inverse matrix is found to be

$$(g_{ij}^{-1}) = \begin{pmatrix} g_{11}^{-1} & g_{12}^{-1} & g_{13}^{-1} & 0 & 0 \\ g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} & 0 & 0 \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} & 0 & 0 \\ 0 & 0 & 0 & g_{44}^{-1} & 0 \\ 0 & 0 & 0 & 0 & g_{55}^{-1} \end{pmatrix}, \tag{22}$$

with

$$g_{11}^{-1} = \frac{B}{\sin^2\Theta}\left(\frac{\cos^2\psi}{J_1} + \frac{\sin^2\psi}{J_2}\right), \tag{23}$$

$$g_{12}^{-1} = -B\left(\frac{1}{J_1} - \frac{1}{J_2}\right)\frac{\sin\psi\cos\psi}{\sin\Theta}, \tag{24}$$

$$g_{13}^{-1} = -B\left(\frac{\cos^2\psi}{J_1} + \frac{\sin^2\psi}{J_2}\right)\frac{\cot\Theta}{\sin\Theta}, \qquad g_{22}^{-1} = B\left(\frac{\sin^2\psi}{J_1} + \frac{\cos^2\psi}{J_2}\right), \tag{25}$$

$$g_{23}^{-1} = B\left(\frac{1}{J_1} - \frac{1}{J_2}\right)\cot\Theta\sin\psi\cos\psi, \tag{26}$$

$$g_{33}^{-1} = B\left(\frac{\cos^2\psi}{J_1} + \frac{\sin^2\psi}{J_2}\right)\cot^2\Theta + \frac{B}{J_3}, \qquad g_{44}^{-1} = 1, \qquad g_{55}^{-1} = \frac{1}{\beta^2}. \tag{27}$$

Using these matrix elements and the value of the determinant from Eq. (20) in the usual Pauli–Podolsky prescription [10]

$$(\nabla\Phi)^i = g^{ij}\frac{\partial\Phi}{\partial x^j}, \qquad \nabla^2\Phi = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j\Phi, \tag{28}$$

we obtain

$$T = -\frac{\hbar^2}{2B}\nabla^2 = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}\right.$$

$$\left. -\frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right], \tag{29}$$

where Q_k are the components of the angular momentum in the intrinsic frame

$$Q_x = -i\left(-\frac{\cos\psi}{\sin\Theta}\frac{\partial}{\partial\Phi} + \sin\psi\frac{\partial}{\partial\Theta} + \cot\Theta\cos\psi\frac{\partial}{\partial\psi}\right), \tag{30}$$

$$Q_y = -i\left(-\frac{\sin\psi}{\sin\Theta}\frac{\partial}{\partial\Phi} + \cos\psi\frac{\partial}{\partial\Theta} - \cot\Theta\sin\psi\frac{\partial}{\partial\psi}\right), \quad Q_z = -i\frac{\partial}{\partial\psi}. \tag{31}$$

The connection between the position-dependent effective mass and curved spaces has been considered in Ref. [8]. According to the findings of Ref. [8], one expects in the present case all elements of the matrix (15) to be divided by f^2

$$g'_{ij} = \frac{g_{ij}}{f^2}. \tag{32}$$

As a result, the determinant of the matrix will be

$$g' = \frac{g}{f^{10}}, \tag{33}$$

and the volume element will be

$$dV' = \frac{dV}{f^5}. \tag{34}$$

The elements of the inverse matrix will be

$$g'^{-1}_{ij} = f^2 g^{-1}_{ij}. \tag{35}$$

In Ref. [11] the Bohr equation with a mass depending on the deformation

$$B(\beta) = \frac{B_0}{(f(\beta))^2}, \tag{36}$$

where B_0 is a constant, has been considered, leading to a modified Bohr equation of the form

$$H\Psi = \left[-\frac{1}{2}\frac{\sqrt{f}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 f \frac{\partial}{\partial\beta}\sqrt{f} - \frac{f^2}{2\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma \frac{\partial}{\partial\gamma} \right.$$

$$\left. + \frac{f^2}{8\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)} + v_{eff} \right] \Psi = \epsilon\Psi, \tag{37}$$

where reduced energies $\epsilon = B_0 E/\hbar^2$ and reduced potentials $v = B_0 V/\hbar^2$ have been used, with

$$v_{eff} = v(\beta,\gamma) + \frac{1}{4}(1-\delta-\lambda)f\nabla^2 f + \frac{1}{2}\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(\nabla f)^2. \tag{38}$$

According to Ref. [8], in order to obtain the Schrödinger equation in the form of Eq. (37), one has to start with the equation

$$H_g\tilde{\Psi} = \left[-\frac{1}{2}\nabla^2 + u_g \right]\tilde{\Psi} = \left[-\frac{1}{2}\frac{1}{\sqrt{g'}}\partial_i \sqrt{g'} g'^{-1}_{ij}\partial_j + u_g \right]\tilde{\Psi}, \tag{39}$$

where

$$\tilde{\Psi} = f^{5/2}\Psi, \tag{40}$$

while reduced energies and reduced potentials are used, as in Eq. (37). The exponent in the last equation is related to the dimensionality of the space.

Substituting the g' matrix elements and determinant in Eq. (39), and performing the relevant calculation (which closely resembles the pure Bohr case, except for the 44-term), we see that Eqs. (39) and (37) do coincide with

$$u_g = u_{eff} + f\ddot{f} - 2(\dot{f})^2 + 4\frac{f\dot{f}}{\beta}, \quad \dot{f} = \frac{df}{d\beta}, \quad \ddot{f} = \frac{d^2 f}{d\beta^2}. \tag{41}$$

This result has several important consequences.

1) It becomes clear that solving the Schrödinger equation (37) with deformation dependent mass is equivalent to solving a modified Bohr equation (39) with different metric matrix g' and another effective potential, u_g. Between the two equivalent schemes, one chooses to solve Eq. (37) instead of Eq. (39), just because the former can be solved analytically through the use of SUSYQM techniques.

2) The wave functions $\tilde{\Psi} = f^{5/2}\Psi$ are accompanied by the volume element $dV' = dV/f^5$. As a result

$$\int \tilde{\Psi}^* \tilde{\Psi} dV' = \int (f^{5/2}\Psi^*)(f^{5/2}\Psi)\frac{dV}{f^5} = \int \Psi^* \Psi dV, \tag{42}$$

i.e., the wave functions Ψ of the deformation dependent mass problem correspond to the usual Bohr volume element dV.

3) The simple relation between $\tilde{\Psi}$ and Ψ also shows that the wave functions Ψ satisfy the well-known 24 symmetries of Bohr wave functions [9], which the wave functions $\tilde{\Psi}$ satisfy by construction. If these symmetries were not satisfied, the solutions could not have been used for the description of nuclei.

The solution of Eq. (37) using SUSYQM techniques [11] is described in this conference by P. E. Georgoudis [12].

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