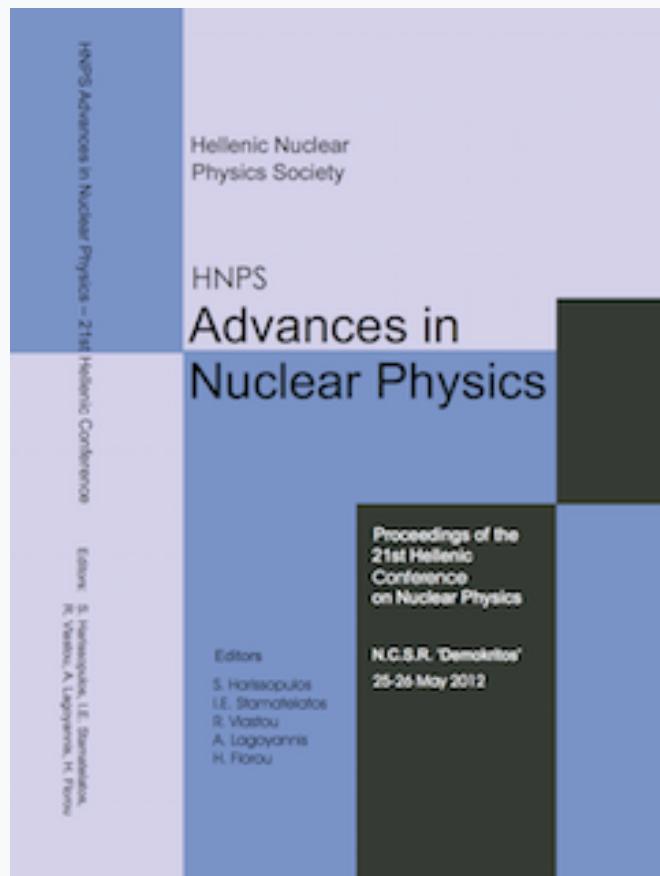


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# Nuclear equation of state effects on the r-mode instability of neutron stars

Ch.C. Moustakidis

*Department of Theoretical Physics, Aristotle University of Thessaloniki, 54124  
Thessaloniki, Greece*

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## Abstract

We study the effect of nuclear equation of state on the r-mode instability of a rotating neutron star. We consider the case where the crust of the neutron star is perfectly rigid and we employ the related theory. The effects of the density dependence of the nuclear symmetry energy on r-mode instability properties are presented and analyzed. A comparison of theoretical predictions with observed neutron stars in low-mass X-ray binaries is also performed and analyzed.

*Key words:* Neutron stars; Nuclear equation of state; Nuclear symmetry energy; r-mode instability; Gravitational waves.

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## 1 Introduction

The oscillations and instabilities of relativistic stars gained a lot of interest in the last decades because of the possible detection of their gravitational waves [1,2]. Especially neutron stars may suffer a number of instabilities which come in different flavors but they have a general feature in common, they can be directly associated with unstable modes of oscillation. In the present work we concentrate our study on the called r-mode instability. The r-modes are oscillations of rotating stars whose restoring force is the Coriolis force. The gravitational radiation-driven instability of these modes has been proposed as an explanation for the observed relatively low spin frequencies of young neutron stars and of accreting neutron stars in low-mass X-ray binaries as well [1]. This instability can only occur when the gravitational-radiation driving time scale of the r-mode is shorter than the time scales of the various dissipation mechanisms that may occur in the interior of the neutron star.

The motivation of the present work, in view of the above studies, is to study extensively the nuclear equation of state (EOS) effect on the r-mode instability

of neutron stars with a perfectly rigid crust [1]. Actually EOS affects the time scales associated with the r-mode, in two different ways. Firstly, EOS defines the radial dependence of the mass density distribution  $\rho(r)$ , which is the basic ingredient of the relevant integrals. Secondly, it defines the core-crust transition density  $\rho_c$  and also the core radius  $R_c$  which is the upper limit of the mentioned integrals.

## 2 Stability of the r-modes

The *r*-modes evolve with time dependence  $e^{i\omega t - t/\tau}$  as a consequence of ordinary hydrodynamics and the influence of the various dissipative processes. The real part of the frequency of these modes,  $\omega$ , is given by

$$\omega = -\frac{(m-1)(m+3)}{m+1}\Omega, \quad (1)$$

where  $\Omega$  is the angular velocity of the unperturbed star. The imaginary part  $1/\tau$  is determined by the effects of gravitational radiation, viscosity, etc. In the small-amplitude limit, a mode is a driven, damped harmonic oscillator with an exponential damping time scale

$$\frac{1}{\tau(\Omega, T)} = \frac{1}{\tau_{GR}(\Omega)} + \frac{1}{\tau_{bv}(\Omega, T)} + \frac{1}{\tau_v(\Omega, T)}, \quad (2)$$

where  $\tau_{GR}$ ,  $\tau_{bv}$ ,  $\tau_v$  are gravitational radiation, bulk viscosity and shear viscosity times scales. Gravitational radiation tends to drive the *r*-modes unstable, while viscosity suppresses the instability. More precisely dissipative effects cause the mode to decay exponentially as  $e^{-t/\tau}$  (i.e., the mode is stable) as long as  $\tau > 0$ . The damping time scale due to viscous dissipation at the boundary layer of the perfectly rigid crust and fluid core is given by [1]

$$\tau_v = \frac{1}{2\Omega} \frac{2^{m+3/2}(m+1)!}{m(2m+1)!! I_m} \sqrt{\frac{2\Omega R_c^2 \rho_c}{\eta_c}} \int_0^{R_c} \frac{\rho(r)}{\rho_c} \left(\frac{r}{R_c}\right)^{2m+2} \frac{dr}{R_c}, \quad (3)$$

where the quantities  $R_c$ ,  $\rho_c$  and  $\eta_c$  are the radius, density and viscosity of the fluid at the outer edge of the core. The fiducial viscous time scale  $\tilde{\tau}_v$  is defined as

$$\tau_v = \tilde{\tau}_v \left(\frac{\Omega_0}{\Omega}\right)^{1/2} \left(\frac{T}{10^8 K}\right), \quad (4)$$

where  $\Omega_0 = \sqrt{\pi G \bar{\rho}}$  and  $\bar{\rho} = 3M/4\pi R^3$  is the mean density of the star.

The gravitational radiation time scale is given by [1]

$$\frac{1}{\tau_{GR}} = -\frac{32\pi G\Omega^{2m+2}}{c^{2m+3}} \frac{(m-1)^{2m}}{[(2m+1)!]^2} \left(\frac{m+2}{m+1}\right)^{2m+2} \int_0^{R_c} \rho(r) r^{2m+2} dr, \quad (5)$$

while the fiducial gravitational radiation time scale  $\tilde{\tau}_{GR}$  is defined as

$$\tau_{GR} = \tilde{\tau}_{GR} \left(\frac{\Omega_0}{\Omega}\right)^{2m+2}. \quad (6)$$

The critical angular velocity  $\Omega_c$ , above which the  $r$ -mode is unstable, is defined by the condition  $\tau_{GR} = \tau_v$  and is given, for  $m = 2$ , by [1]

$$\frac{\Omega_c}{\Omega_0} = \left(\frac{\tilde{\tau}_{GR}}{\tilde{\tau}_v}\right)^{2/11} \left(\frac{10^8 K}{T}\right)^{2/11}. \quad (7)$$

Moreover, the maximum angular velocity  $\Omega_K$  (Kepler angular velocity) for any star occurs when the material at the surface effectively orbits the star. This velocity is nearly  $\Omega_K = \frac{2}{3}\Omega_0$ . Thus, there is a critical temperature below which the gravitational radiation instability is completely suppressed by viscosity and is given by [1]

$$\frac{T_c}{10^8 K} = \left(\frac{\Omega_0}{\Omega_K}\right)^{11/2} \left(\frac{\tilde{\tau}_{GR}}{\tilde{\tau}_v}\right) = \left(\frac{3}{2}\right)^{11/2} \left(\frac{\tilde{\tau}_{GR}}{\tilde{\tau}_v}\right). \quad (8)$$

Employing Eqs. (7) and (8) the critical angular velocity is expressed in terms of  $T_c$ , that is

$$\frac{\Omega_c}{\Omega_0} = \frac{\Omega_K}{\Omega_0} \left(\frac{T_c}{T}\right)^{2/11} = \frac{2}{3} \left(\frac{T_c}{T}\right)^{2/11}. \quad (9)$$

Once the equation of state for the neutron star core and crust is fixed, then all the ingredients of the  $r$ -mode instability, that is the transition density  $\rho_c$ , the radial dependence of the mass density  $\rho(r)$ , and the global properties of the neutron star (mass, radius and core radius) are determined in a self-consistent way.

### 3 The nuclear model

The model used here, which has already been presented and analyzed in previous papers [3], is designed to reproduce the results of the microscopic cal-

culations of both nuclear and neutron-rich matter at zero temperature and can be extended to finite temperature [3]. The energy per baryon at  $T = 0$ , is given by

$$\begin{aligned}
E_b(n, I) = & \frac{3}{10} E_F^0 u^{2/3} \left[ (1 + I)^{5/3} + (1 - I)^{5/3} \right] \\
& + \frac{1}{3} A \left[ \frac{3}{2} - \left( \frac{1}{2} + x_0 \right) I^2 \right] u \frac{\frac{2}{3} B \left[ \frac{3}{2} - \left( \frac{1}{2} + x_3 \right) I^2 \right] u^\sigma}{1 + \frac{2}{3} B' \left[ \frac{3}{2} - \left( \frac{1}{2} + x_3 \right) I^2 \right] u^{\sigma-1}} \\
& + \frac{3}{2} \sum_{i=1,2} \left[ C_i + \frac{C_i - 8Z_i}{5} I \right] \left( \frac{\Lambda_i}{k_F^0} \right)^3 \left( \frac{((1 + I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}} - \tan^{-1} \frac{((1 + I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}} \right) \\
& + \frac{3}{2} \sum_{i=1,2} \left[ C_i - \frac{C_i - 8Z_i}{5} I \right] \left( \frac{\Lambda_i}{k_F^0} \right)^3 \left( \frac{((1 - I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}} - \tan^{-1} \frac{((1 - I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}} \right).
\end{aligned} \tag{10}$$

In Eq. (10),  $I$  is the asymmetry parameter ( $I = (n_n - n_p)/n$ ) and  $u = n/n_0$ , with  $n_0$  denoting the equilibrium symmetric nuclear matter density,  $n_0 = 0.16 \text{ fm}^{-3}$ . The parameters  $A$ ,  $B$ ,  $\sigma$ ,  $C_1$ ,  $C_2$  and  $B'$  which appear in the description of symmetric nuclear matter are determined in order that  $E_b(n = n_0, I = 0) = -16 \text{ MeV}$ ,  $n_0 = 0.16 \text{ fm}^{-3}$ , and the incompressibility is  $K = 240 \text{ MeV}$  and have the values  $A = -46.65$ ,  $B = 39.45$ ,  $\sigma = 1.663$ ,  $C_1 = -83.84$ ,  $C_2 = 23$  and  $B' = 0.3$ . The finite range parameters are  $\Lambda_1 = 1.5k_F^0$  and  $\Lambda_2 = 3k_F^0$  and  $k_F^0$  is the Fermi momentum at the saturation point  $n_0$ . The baryon energy is written also as a function of the baryon density  $n$  and the proton fraction  $x$  ( $x = n_p/n$ ), that is  $E_b(n, x)$ , by replacing  $I = 1 - 2x$ . The additional parameters  $x_0$ ,  $x_3$ ,  $Z_1$ , and  $Z_2$  employed to determine the properties of asymmetric nuclear matter are treated as parameters constrained by empirical knowledge. The parameterizations used in the present model have only a modest microscopic foundation. Nonetheless, they have the merit of being able to closely approximate more physically motivated calculations.

The energy  $E_b(n, I)$  can be expanded around  $I = 0$  as follows

$$\begin{aligned}
E_b(n, I) = & E_b(n, I = 0) + E_{sym,2}(n)I^2 + E_{sym,4}(n)I^4 + \dots \\
& + E_{sym,2k}(n)I^{2k} + \dots,
\end{aligned} \tag{11}$$

where the coefficients of the expansion are given by the expression

$$E_{sym,2k}(n) = \frac{1}{(2k)!} \frac{\partial^{2k} E_b(n, I)}{\partial I^{2k}} \bigg|_{I=0}. \tag{12}$$

In (11), only even powers of  $I$  appear due to the fact that the strong interaction must be symmetric under exchange of neutrons with protons i.e. the

contribution to the energy must be independent of the sign of the difference  $n_n - n_p$ . The nuclear symmetry energy  $E_{sym}(n)$  is defined as the coefficient of the quadratic term, that is

$$E_{sym}(n) = E_{sym,2}(n) = \frac{1}{2!} \frac{\partial^2 E_b(n, I)}{\partial I^2} \Big|_{I=0}. \quad (13)$$

The slope of the symmetry energy  $L$  at nuclear saturation density  $n_0$ , which is correlated with the crust-core transition density  $n_t$  in a neutron star, is defined as

$$L = 3n_0 \frac{\partial E_{sym}(n)}{\partial n} \Big|_{n=n_0}. \quad (14)$$

By suitably choosing the parameters  $x_0$ ,  $x_3$ ,  $Z_1$ , and  $Z_2$ , it is possible to obtain different forms for the density dependence of the symmetry energy  $E_{sym}(n)$  as well as on the value of the slope parameter  $L$ . We take as a range of  $L$   $50 \text{ MeV} \leq L \leq 110 \text{ MeV}$  where the value of the symmetry energy at saturation density is fixed to be  $E_{sym}(n_0) = 30 \text{ MeV}$ . Actually, for each value of  $L$  the density dependence of the symmetry energy is adjusted so that the energy of pure neutron matter to be comparable with those of existing state-of-the-art calculations.

The total pressure  $P(n, x)$ , in the core of a neutron star, is decomposed into baryon and electron contributions

$$P(n, x) = P_b(n, x) + P_e(n, x), \quad (15)$$

where  $P_b(n, x) = n^2 \frac{\partial E_b(n, x)}{\partial n}$ . The electrons are considered as a non-interacting relativistic Fermi gas and their contribution to the total energy density  $\epsilon_e(n, x)$  and pressure  $P_e(n, x)$  reads

$$\epsilon_e(n, x) = \frac{\hbar c}{4\pi^2} (3\pi^2 xn)^{4/3}, \quad P_e(n, x) = \frac{\hbar c}{12\pi^2} (3\pi^2 xn)^{4/3}. \quad (16)$$

Now the total energy density  $\epsilon_{tot}$  and pressure  $P_{tot}$  of charge neutral and chemically equilibrium nuclear matter is

$$\epsilon_{tot} = \epsilon_b + \epsilon_e, \quad (17)$$

$$P_{tot} = P_b + P_e. \quad (18)$$

From Eqs. (17) and (18) we construct the equation of state in the form  $\epsilon = \epsilon(P)$ .

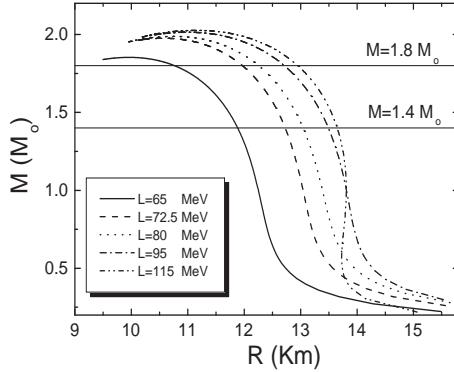


Fig. 1. The mass-radius relations for the selected EOSs

#### 4 Results and Discussion

We employ a phenomenological model for the energy per baryon of the asymmetric nuclear matter. By suitably choosing the parametrization of the model we obtain various forms for the density dependence of the energy per baryon of neutron matter. In Fig. 1 we display the mass-radius relation for neutron stars for the selected equation of states. All of them predict maximum mass neutron stars even higher than  $1.8M_{\odot}$ . The case where  $L = 50$  MeV has been excluded since it leads to very low value for the maximum mass of a neutron star.

In Fig. 2 are displayed the r-mode instability windows for neutron stars with mass  $1.4M_{\odot}$  and  $1.8M_{\odot}$  for the selected equations of state as a function of the temperature. For low values of temperature ( $T \leq 10^9$  K) we plot the ratio

$$\frac{\Omega_c}{\Omega_0} = \left( \frac{\tilde{\tau}_{GR}}{\tilde{\tau}_{ee}} \right)^{2/11} \left( \frac{10^8 K}{T} \right)^{2/11}, \quad (19)$$

while for  $T \geq 10^9$  K we plot the ratio

$$\frac{\Omega_c}{\Omega_0} = \left( \frac{\tilde{\tau}_{GR}}{\tilde{\tau}_{nn}} \right)^{2/11} \left( \frac{10^8 K}{T} \right)^{2/11}. \quad (20)$$

The most striking feature is the location of the ratio  $\Omega_c/\Omega_0$  in a narrow interval (mainly in the case of neutron star with mass  $1.4M_{\odot}$ ). Actually, the ratio  $\Omega_c/\Omega_0$  increases around 7% (for  $T \leq 10^9$  K) and around 9% (for  $T \geq 10^9$  K) with the lower values corresponding to the case of  $L = 65$  MeV and the higher to the case  $L = 80$  MeV. It is concluded that the values of the ratio saturate for  $L$  close to the value 80 MeV. In the case of a neutron star with

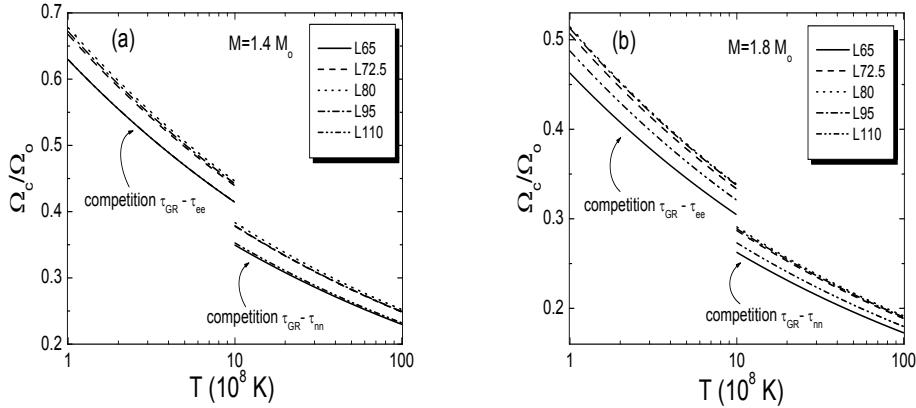


Fig. 2. Temperature dependence of the critical angular velocity ratio  $\Omega_c/\Omega_0$  for a neutron star with mass  $M = 1.4M_\odot$  (a) and  $M = 1.8M_\odot$  (b) constructed for the selected EOSs.

mass  $M = 1.8M_\odot$  the ratio  $\Omega_c/\Omega_0$  increases around 10% both for  $T \leq 10^9$  K and for  $T \geq 10^9$  K with the lower values corresponding to the case of  $L = 65$  MeV and the higher to the case  $L = 80$  MeV.

In Fig. 3 we compare the r-mode instability window for the five selected equation of states with those of the observed neutron stars in low-mass x-ray binaries (LMXBs) for  $M = 1.4M_\odot$  and  $M = 1.8M_\odot$ . Firstly, we found that the instability window drops by  $\simeq 20 - 40$  Hz when the mass is raised from  $M = 1.4M_\odot$  to  $M = 1.8M_\odot$ . We examine four cases of LMXBs that is the 4U 1608-522 at 620 Hz, 4U 1636-536 at 581 Hz, MXB 1658-298 at 567 Hz and EXO 0748-676 at 552 Hz [4,5]. The masses of the mentioned stars are not measured accurately but the core temperature  $T$  is derived from their observed accretion luminosity. It is obvious from Fig. 3 that for a  $M = 1.8M_\odot$  all the considered LMXBs lie inside instability window. The case is similar for a  $M = 1.4M_\odot$  where the only exception the EXO 0748-676. The LMXBs should be out of the instability window. Consequently, one can presume that either the LMXBs masses are even lower than  $M = 1.4M_\odot$  or the softer equation of state is more preferred. However, additional theoretical and observation work must be dedicated before a definite conclusion.

Summarizing, we considerer the effects, on r-mode instability, due to the presence of a solid crust in a neutron star. By employing a phenomenological nuclear model we calculated the equation of state of  $\beta$ -stable matter which characterizes the neutron star core and defines the transition density between the liquid core and the solid crust. The stiffness of the equation of state parameterized via the slope parameter  $L$ . We found also that the instability window drops by  $\simeq 20 - 40$  Hz when the mass of a neutron star is raised from  $M = 1.4M_\odot$  to  $M = 1.8M_\odot$ . Finally we compared the r-mode instability window for the five selected equation of states with those of the observed neutron stars in low-mass x-ray binaries (LMXBs) for  $M = 1.4M_\odot$  and  $M = 1.8M_\odot$ .

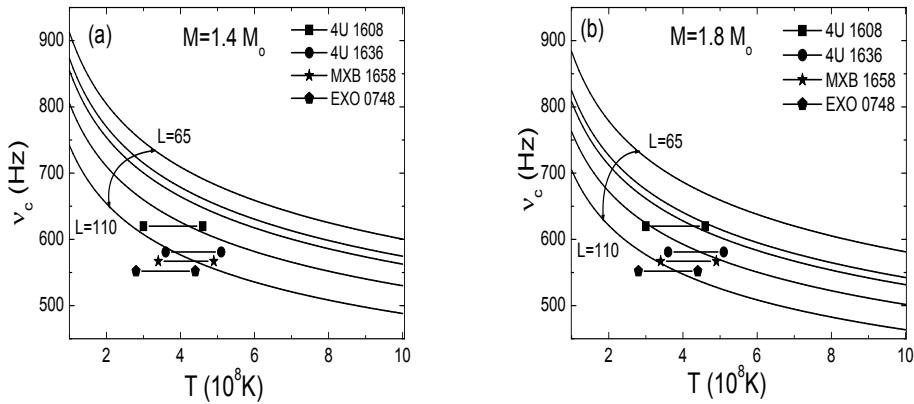


Fig. 3. The critical frequency temperature dependence for a neutron star with mass  $M = 1.4M_{\odot}$  (a) and  $M = 1.8M_{\odot}$  (b) constructed for the selected EOSs. In addition, the location of the observed short-recurrence-time LMXBs [4,5]

According to our finding one can presume that either the LMXBs masses are even lower than  $M = 1.4M_{\odot}$  or a softer equation of state is more preferred.

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