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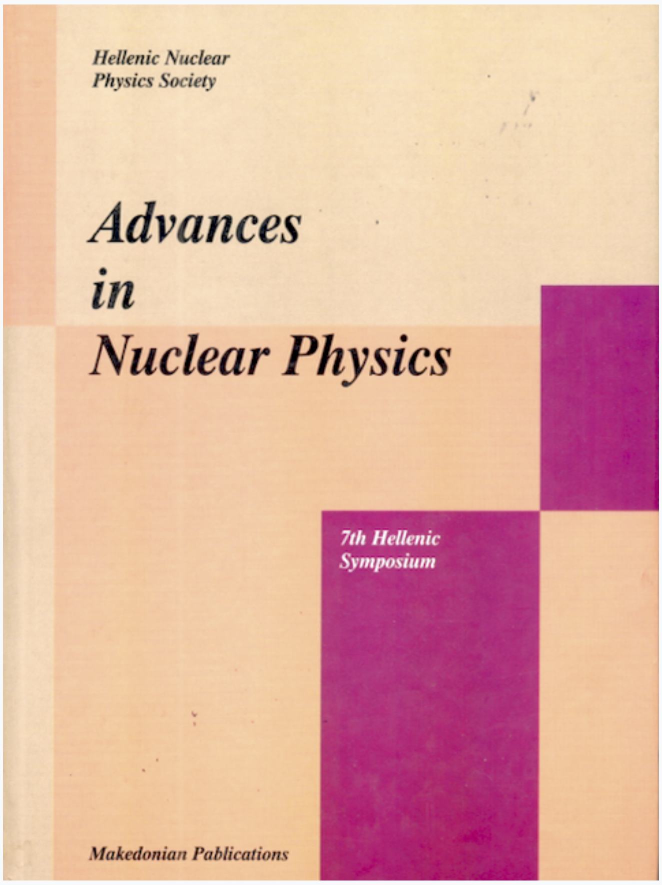

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Symmetries of Anisotropic Harmonic Oscillators with Rational Ratios of Frequencies and their Relations to U(N) and O(N+1)

Dennis Bonatsos[a,b], C. Daskaloyannis[c], P. Kolokotronis[a] and D. Lenis[a]

[a] *Institute of Nuclear Physics, N.C.S.R. "Demokritos", GR-15310 Aghia Paraskevi, Attiki, Greece*

[b] *European Centre for Theoretical Research in Nuclear Physics and Related Areas (ECT*), Villa Tambosi, Strada delle Tabarelle 286, I-38050 Villazzano (Trento), Italy*

[c] *Department of Physics, Aristotle University of Thessaloniki, GR-54006 Thessaloniki, Greece*

Abstract

The concept of bisection of a harmonic oscillator or hydrogen atom, used in the past in establishing the connection between U(3) and O(4), is generalized into multisection (trisection, tetrasection, etc). It is then shown that all symmetries of the N-dimensional anisotropic harmonic oscillator with rational ratios of frequencies (RHO), some of which are underlying the structure of superdeformed and hyperdeformed nuclei, can be obtained from the U(N) symmetry of the corresponding isotropic oscillator with an appropriate combination of multisections. Furthermore, it is seen that bisections of the N-dimensional hydrogen atom, which possesses an O(N+1) symmetry, lead to the U(N) symmetry, so that further multisections of the hydrogen atom lead to the symmetries of the N-dim RHO. The opposite is in general not true, i.e. multisections of U(N) do not lead to O(N+1) symmetries, the only exception being the occurence of O(4) after the bisection of U(3).

1 Introduction

Anisotropic harmonic oscillators with rational ratios of frequencies (RHOs) [1–7] of current interest in several branches of physics. Their symmetries form the basis for the understanding [8–12] of the occurence of superdeformed and hyperdeformed nuclear shapes [13,14] at very high angular momenta. In addition, they have been recently connected [15,16] to the underlying geometrical structure in the Bloch–Brink α-cluster model [17]. They are also becoming of interest for the interpretation of the observed shell structure in atomic clusters [18], especially after the realization that large deformations can occur in such systems [19]. An interesting problem is to what extend the various symmetries of the RHOs, occuring for different frequency ratios, are related to other known symmetries. A well-known example is the case of the 3-dimensional RHO with frequency ratios 2:2:1, which is known to possess the O(4) symmetry [20].

In this paper we show how the symmetries of the N-dim RHO can be obtained from the U(N) symmetry of the corresponding isotropic harmonic oscillator (HO) by appropriate symmetry operations, namely multisections, which are generalizations of the concept of bisection, introduced in [20]. It will furthermore be shown that these symmetries can also be obtained from the O(N+1) symmetry of the N-dim hydrogen atom, since a bisection leads from O(N+1) to U(N), so that further multisections lead to RHO symmetries. However, despite the fact that the N-dim RHO symmetries can be obtained from the O(N+1) symmetry by appropriate multisections, they are not orthogonal symmetries themselves (with the exception of 2:2:1 mentioned above).

In Section 2 of this paper multisections of the N-dim harmonic oscillator are defined and used in obtaining the symmetries of the various RHOs. A similar procedure is followed in Section 3 for the N-dim hydrogen atom. Section 4 contains discussion of the present results and implications for further work.

2 Multisections of the harmonic oscillator

The Hamiltonian of the N-dim RHO reads

$$H = \frac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + \frac{x_k^2}{m_k^2}\right), \tag{1}$$

where m_i are natural numbers prime to each other. The energy eigenvalues are given by

$$E = \sum_{k=1}^{N}\frac{1}{m_k}\left(n_k + \frac{1}{2}\right), \tag{2}$$

where n_k is the number of quanta in the k-th direction. Alternatively, the energy eigenvalues can be written as [21,22]

$$E = \Sigma + \sum_{k=1}^{N}\frac{2q_k - 1}{2m_k}, \tag{3}$$

with $q_k = 1, 2, \ldots, m_k$, the connection between the two pictures been given by

$$n_k = [n_k/m_k]m_k + \mathrm{mod}(n_k, m_k), \tag{4}$$

$$\Sigma = \sum_{k=1}^{N} [n_k/m_k], \tag{5}$$

$$q_k = \mathrm{mod}(n_k, m_k) + 1, \tag{6}$$

where $[x]$ stands for the integer part of x.

2.1 The 3-dimensional oscillator

Let us consider the completely symmetric irreps of U(3), $[N00]$, the dimensions of which are given by

$$d(N) = \frac{(N+1)(N+2)}{2}, \qquad N = 0, 1, 2, \ldots \tag{7}$$

Using the Cartesian notation (n_x, n_y, n_z) for the U(3) states, as in [20], we have the following list:

N=0: (000)

N=1: (100) (010) (001)

N=2: (200) (020) (002) (110) (101) (011)

N=3: (300) (030) (003) (210) (120) (201) (102) (021) (012) (111)

N=4: (400) (040) (004) (310) (130) (301) (103) (031) (013) (220) (202) (022) (211) (121) (112)

N=5: (500) (050) (005) (410) (140) (401) (104) (041) (014) (320) (230) (302) (203) (032) (023) (311) (131) (113) (221) (212) (122).

We see that the corresponding degeneracies are 1, 3, 6, 10, 15, 21, ..., which correspond to the dimensions of the U(3) irreps, as mentioned above, i.e. to the 1:1:1 HO.

Choosing only the n_z=odd states, one is left with the following list

N=1: (001)

N=2: (101) (011)

N=3: (003) (201) (021) (111)

N=4: (301) (103) (031) (013) (211) (121)

N=5: (005) (401) (041) (203) (023) (311) (131) (113) (221), while choosing only the n_z=even states, one is left with the following list:

N=0: (000)

N=1: (100) (010)

N=2: (200) (020) (002) (110)

N=3: (300) (030) (210) (120) (102) (012)

N=4: (400) (040) (004) (310) (130) (220) (202) (022) (112)

N=5: (500) (050) (410) (140) (104) (014) (320) (230) (302) (032) (212) (122).

We see that in both cases the degeneracies are 1, 2, 4, 6, 9, 12, 16, 20, ..., which are the degeneracies of the 3-dim RHO with ratios 2:2:1. Therefore a **bisection** of the 1:1:1 RHO states, distinguishing states with $\mathrm{mod}(n_z, 2) = 0$ and states with $\mathrm{mod}(n_z, 2) = 1$, results it two interleaving 2:2:1 sets of levels.

By analogy, a **trisection** can be made by distinguishing states with $\mathrm{mod}(n_z, 3) = 0$ or $\mathrm{mod}(n_z, 3) = 1$ or $\mathrm{mod}(n_z, 3) = 2$. For $\mathrm{mod}(n_z, 3) = 0$ we obtain

N=0: (000)

N=1: (100) (010)

N=2: (200) (020) (110)

N=3: (300) (030) (003) (210) (120)

N=4: (400) (040) (310) (130) (103) (013) (220)

N=5: (500) (050) (410) (140) (320) (230) (203) (023) (113),

while for $\mathrm{mod}(n_z, 3) = 1$ one has

N=1: (001)

N=2: (101) (011)

N=3: (201) (021) (111)

N=4: (004) (301) (031) (211) (121)

N=5: (401) (104) (041) (014) (311) (131) (221),

and for $\mathrm{mod}(n_z, 3) = 2$ one has

N=2: (002)

N=3: (102) (012)

N=4: (202) (022) (112)

N=5: (005) (302) (032) (212) (122).

The degeneracies obtained are 1, 2, 3, 5, 7, 9, 12, 15, 18, ..., which correspond to the 3:3:1 RHO. Therefore a trisection of the 1:1:1 HO results in three interleaving sets of 3:3:1 RHO states.

Similarly a **tetrasection** is defined by selecting states with $\mathrm{mod}(n_z, 4) = 0$, or 1, or 2, or 3. In the case of $\mathrm{mod}(n_z, 4) = 0$ one has

N=0: (000)

N=1: (100) (010)

N=2: (200) (020) (110)

N=3: (300) (030) (210) (120)

N=4: (400) (040) (004) (310) (130) (220)

N=5: (500) (050) (410) (140) (104) (014) (320) (230),

while for $\mathrm{mod}(n_z, 4) = 1$ one obtains

N=1: (001)

N=2: (101) (011)

N=3: (201) (021) (111)

N=4: (301) (031) (211) (121)

N=5: (005) (401) (041) (311) (131) (221),

for $\mathrm{mod}(n_z, 4) = 2$ one has

N=2: (002)

N=3: (102) (012)

N=4: (202) (022) (112)

N=5: (302) (032) (212) (122),

and for mod$(n_z, 4) = 3$ one gets

N=3: (003)

N=4: (103) (013)

N=5: (203) (023) (113) .

The degeneracies obtained are 1, 2, 3, 4, 6, 8, 10, 12, 15, 18, ..., which characterize the 4:4:1 RHO. Therefore a tetrasection of the 1:1:1 HO leads to four interleaving sets of 4:4:1 RHO states.

In general, an *n-section* of the 1:1:1 HO is obtained by separating states with $mod(n_z, n)$ =0, or 1, or 2, ..., or $n-1$. In this case n interleaving sets of the n:n:1 RHO states, which corresponds to an oblate shape, are obtained. It is clear that n-sections using n_x or n_y instead of n_z lead to the same conclusions.

One can consider successively more than one bisections, trisections, etc. Let us consider more than one bisections first.

Getting the results of the mod$(n_z, 2) = 0$ bisection of the HO and applying a mod$(n_y, 2)$ =0 bisection on them we obtain

N=0: (000)

N=1: (100)

N=2: (200) (020) (002)

N=3: (300) (120) (102)

N=4: (400) (040) (004) (220) (202) (022)

N=5: (500) (140) (104) (320) (302) (122).

The degeneracy pattern is 1, 1, 3, 3, 6, 6, 10, 10, i.e. "two copies" of the 1:1:1 degeneracies, which corresponds to the 2:1:1 RHO. The same result is obtained for any combination of two bisections along two differerent axes.

Bisecting the 1:1:1 HO for a third time, along the x-axis this time by using mod$(n_x, 2)$ =0, one obtains

N=0: (000)

N=2: (200) (020) (002)

N=4: (400) (040) (004) (220) (202) (022).

Table 1

Degeneracies of various 3-dim anisotropic harmonic oscillators with rational ratios of frequencies (RHOs) obtained from the U(3) symmetry of the isotropic 3-dim harmonic oscillator (HO) by the application of various multisections. The first line corresponds to the isotropic 3-dim HO. In the rest of the lines the first column contains the appropriate multisection, while the second column contains the frequency ratios $m_1 : m_2 : m_3$ of the resulting RHO.

U(3)	1:1:1	1	3	6	10	15	21	28	36				
1 bisection	2:2:1	1	2	4	6	9	12	16	20				
1 trisection	3:3:1	1	2	3	5	7	9	12	15	18	22	26	
1 tetrasection	4:4:1	1	2	3	4	6	8	10	12	15	18	21	
2 bisections	1:1:2	1	1	3	3	6	6	10	10	15	15	21	
2 trisections	1:1:3	1	1	1	3	3	3	6	6	6	10	10	
2 tetrasections	1:1:4	1	1	1	1	3	3	3	3	6	6		
3 bisections	1:1:1	1	3	6	10	15	21	28	36				
3 trisections	1:1:1	1	3	6	10	15	21	28	36				
3 tetrasection	1:1:1	1	3	6	10	15	21	28	36				

The degeneracy pattern is 1, 3, 6, 10, ..., i.e. that of the original 1:1:1 HO.

Furthermore one can easily see that:

i) Two trisections along different axes lead to degeneracies 1, 1, 1, 3, 3, 3, 6, 6, 6, ..., i.e. to the 3:1:1 RHO pattern ("three copies" of the 1:1:1 degeneracies).

ii) Three trisections lead to the original 1:1:1 HO degeneracy pattern.

iii) Two tetrasections lead to degeneracies 1, 1, 1, 1, 3, 3, 3, 3, 6, 6, 6, 6, ..., i.e. to the 4:1:1 RHO pattern ("four copies" of the 1:1:1 degeneracies).

iv) Three tetrasections lead back to the original 1:1:1 HO pattern.

The results obtained so far are summarized in Table 1.

In general one can see that:

i) Two n-sections (along different axes) lead to the degeneracy pattern of n:1:1, i.e. to "n copies" of the 1:1:1 degeneracies. n:1:1 corresponds to a prolate shape.

ii) Three n-sections lead back to the degeneracy pattern of the 1:1:1 HO.

One can, of course, apply successive n-sections with different n. For example, applying $\mathrm{mod}(n_z, 2) = 0$, $\mathrm{mod}(n_y, 3) = 0$ and $\mathrm{mod}(n_x, 3) = 0$ one obtains the degeneracy pattern 1, 1, 2, 1, 2, 4, 2, 4, 6, ..., which corresponds to the 2:2:3

Table 2

Same as Table 1, but for the 4-dim oscillator.

U(4)	1:1:1:1	1	4	10	20	35	56				
1 bisection	2:2:2:1	1	3	7	13	22					
2 bisections	2:2:1:1	1	2	5	8	14	20	30	40	55	70
3 bisections	2:1:1:1	1	1	4	4	10	10				
4 bisections	1:1:1:1	1	4	10	20	35	56				
1 trisection	3:3:3:1	1	3	6	11	18	27	35			
2 trisections	3:3:1:1	1	2	3	6	9					
3 trisections	3:1:1:1	1	1	1	4	4	4	10	10	10	20
4 trisections	1:1:1:1	1	4	10	20	35	56				
1 tetrasection	4:4:4:1	1	3	6	10						

oscillator.

In general one can see that by applying a k-section, an l-section and an m-section along different axes one obtains the degeneracy pattern $(kl) : (mk) : (lm)$, where common factors appearing in all three quantities (kl), (mk), (lm) can be dropped out.

We have therefore seen that all the symmetries of the 3-dim RHO can be obtained from the U(3) symmetry of the isotropic 3-dim HO by an appropriate set of n-sections.

A special remark can be made about the 2:2:1 case. The degeneracies obtained there correspond to the dimensions of the irreps of O(4), given by

$$d(\mu_1, \mu_2) = (\mu_1 + \mu_2 + 1)(\mu_1 - \mu_2 + 1). \tag{8}$$

In particular, the degeneracies 1, 4, 9, 16, ... correspond to the integer irreps $(\mu, 0)$ with $\mu = 0, 1, 2, 3, \ldots$, while the degeneracies 2, 6, 12, 20, ... correspond to the spinor irreps $(\frac{n}{2}, \frac{1}{2})$ with $n = 1, 3, 5, 7, \ldots$. This result has been first found by Ravenhall *et al.* [20]. It has been pointed out that O(4) is obtained by imposing a reflection condition on U(3). For example, O(4) is obtained by selecting the states with n_z=odd, a procedure which is equivalent to the insertion of an impenetrable barrier across the xy plane.

2.2 The 4-dimensional oscillator

The relevant information is given in Table 2. The symmetry of the HO in this case is U(4). The first line of the table corresponds to the dimensions of the symmetric irreps of U(4), $[N,0,0,0]$, given by the equation

$$d(N) = \frac{1}{6}(N+1)(N+2)(N+3). \tag{9}$$

Notice that these degeneracies coincide with the dimensions of the symmetric irreps $(N,0)$ of Sp(4). (It is known that Sp(4) is a subalgerbra of U(4).)

One bisection of U(4) leads to the 2:2:2:1 degeneracies.

Two bisections of U(4) lead to the 2:2:1:1 degeneracies 1, 2, 5, 8, 14, 20, 30, 40, 55, 70, ... Out of these, 1, 5, 14, 30, 55, ... correspond to the symmetric irreps $(\mu,0)$ of O(5), while 2, 8, 20, 40, 70, ... correspond to the $(\frac{n}{2},\frac{1}{2})$ irreps of O(5). The relevant formula is:

$$d(m_1,m_2) = \frac{1}{6}(2m_1+3)(2m_2+1)(m_1+m_2+2)(m_1-m_2+1). \tag{10}$$

Notice that the dimensions of the integer irreps (1, 5, 14, 30, 55, ...) are reproduced exactly, while the dimensions of the half-integer irreps are half of the ones given by eq. (10), which are 4, 16, 40, 80, 140, ... , respectively. Therefore the 2:2:1:1 symmetry is not O(5), although it bears certain similarities to it.

The occurence of a symmetry resembling O(5) is not surprising, since U(4) is isomorphic to O(6), which does have an O(5) subalgebra. The generators of the O(5) subalgebra in terms of the U(4) generators have been given explicitly in [23,24] . It is also known that O(5) is isomorphic to Sp(4), which is a subalgebra of U(4), since in general U(2n) possesses an Sp(2n) subalgebra.

Three bisections of U(4) lead to the 2:1:1:1 degeneracies, i.e. to "two copies" of the U(4) degeneracies.

Finally, four bisections of U(4) lead back to the U(4) degeneracies characterizing the 1:1:1:1 HO.

Similarly one can see that a trisection of U(4) leads to the 3:3:3:1 degeneracies, two trisections of U(4) lead to 3:3:1:1, three trisections of U(4) lead to 3:1:1:1, i.e. to "three copies" of the U(4) degeneracies, while four trisections of U(4) lead back to U(4).

2.3 The 5-dimensional oscillator

The relevant results are shown in Table 3. In the first line the dimensions of the symmetric irreps $[N,0,0,0,0]$ of U(5) appear, given by the formula

$$d(N) = \frac{1}{24}(N+1)(N+2)(N+3)(N+4). \tag{11}$$

Table 3

Same as Table 1, but for the 5-dim oscillator.

U(5)	1:1:1:1:1	1	5	15	35	70					
1 bisection	2:2:2:2:1	1	4	11	24	46					
2 bisections	2:2:2:1:1	1	3	8	16	30					
3 bisections	2:2:1:1:1	1	2	6	10	20	30	50	70	105	140
4 bisections	2:1:1:1:1	1	1	5	5	15	15				
5 bisections	1:1:1:1:1	1	5	15	35	70					

There is no symplectic subalgebra in this case.

One bisection of U(5) leads to the 2:2:2:2:1 degeneracies, while two bisections lead to the 2:2:2:1:1 pattern.

Three bisections lead to the 2:2:1:1:1 degeneracies 1, 2, 6, 10, 20, 30, 50, 70, 105, 140, 196, ... In particular, the degeneracies 1, 6, 20, 50, 105, 196, ... correspond to the dimensions of the integer irreps $(N, 0, 0)$ of O(6), while the intermediate degeneracies 2, 10, 30, 70, 140, ... resemble the half-integer irreps $(\frac{n}{2}, \frac{1}{2}, \frac{1}{2})$ of O(6). The relevant formula is

$$d(m_1, m_2, m_3) = \frac{1}{12}(m_1 + m_2 + 3)(m_1 + m_3 + 2)(m_2 + m_3 + 1)$$

$$(m_1 - m_2 + 1)(m_1 - m_3 + 2)(m_2 - m_3 + 1). \tag{12}$$

Again there is a factor of 2 difference for the half-integer irreps: The results in Table 3 are $\frac{1}{2}$ times the results given by the above equation. Therefore the 2:2:1:1:1 symmetry is not an O(6) symmetry.

Finally, four bisections lead to the 2:1:1:1:1 degeneracy pattern, while five bisections lead back to the U(5) degeneracies.

2.4 The 6-dimensional oscillator

In this case the symmetry is U(6). The relevant results are given in Table 4. In the first line, the dimensions of the symmetric irreps $[N, 0, 0, 0, 0, 0]$ of U(6) appear, given by the formula

$$d(N) = \frac{1}{120}(N + 1)(N + 2)(N + 3)(N + 4)(N + 5). \tag{13}$$

They coincide with the dimensions of the symmetric irreps $(N, 0, 0)$ of Sp(6). It is known that Sp(6) is a subalgebra of U(6).

Table 4

Same as Table 1, but for the 6-dim oscillator.

U(6)	1:1:1:1:1:1	1	6	21	56						
1 bisection	2:2:2:2:2:1	1	5	16							
2 bisections	2:2:2:2:1:1	1	4	12							
3 bisections	2:2:2:1:1:1	1	3	9							
4 bisections	2:2:1:1:1:1	1	2	7	12	27	42	77	112	182	252
5 bisections	2:1:1:1:1:1	1	1	6	6	21	21				
6 bisections	1:1:1:1:1:1	1	6	21	56						

One bisection leads to the 2:2:2:2:2:1 degeneracy pattern, while two bisections lead to 2:2:2:2:1:1 and three bisections lead to 2:2:2:1:1:1.

Four bisections lead to degeneracies resembling the dimensions of O(7) irreps. The degeneracies 1, 7, 27, 77, 182, ... correspond to the dimensions of integer O(7) irreps of the form $(m,0,0)$, while the numbers 2, 12, 42, 112, 252, ... resemble the dimensions of the half-integer irreps $(\frac{m}{2},\frac{1}{2},\frac{1}{2})$ of O(7). The relevant formula is

$$d(m_1,m_2,m_3)=\frac{1}{720}(2m_1+5)(2m_2+3)(2m_3+1)$$

$$(m_1+m_2+4)(m_1+m_3+3)(m_2+m_3+2)$$

$$(m_1-m_2+1)(m_1-m_3+2)(m_2-m_3+1). \tag{14}$$

This formula gives for the integer irreps the results of Table 4, but for the half-integer irreps it gives 4 times the results of Table 4. Therefore the 2:2:1:1:1:1 symmetry is not an O(7) symmetry.

2.5 The N-dimensional oscillator

The symmetry is U(N). If N is even, there is an Sp(N) subalgebra, if N is odd there is no such subalgebra.

N bisections lead back to the U(N) irreps.

N−1 bisections lead to the 2:1:1:... :1 symmetry, i.e. to "two copies" of the U(N) irreps.

N−2 bisections lead to the 2:2:1:1:... :1 symmetry, which bears certain similarities to O(N+1). The dimensions of the integer irreps are obtained correctly. The dimensions of the odd irreps differ by a factor of $2^{\nu-1}$, where $\nu=N/2$ for

N even or $\nu = (N-1)/2$ for N odd. Therefore the 2:2:1:1:...:1 symmetry is not in general O(N+1).

N−3 bisections lead to the 2:2:2:1:1:...:1 degeneracies.

Two bisections lead to the 2:2:...:2:1:1 degeneracies.

One bisection leads to the 2:2:...:2:2:1 degeneracies.

Similarly

one n-section leads to the n:n:...:n:n:1 degeneracies,

two n-sections lead to the n:n:...:n:1:1 degeneracies,

N−2 n-sections lead to n:n:1:...:1:1,

N−1 n-sections lead to n:1:1:...:1:1,

N n-sections lead back to U(N).

3 Multisections of the hydrogen atom

So far we have considered multisections of the N-dim harmonic oscillator. We are now going to consider multisections of the hydrogen atom (HA) in N dimensions, which is known to be characterized by the O(N+1) symmetry [25], which is also the symmetry characterizing a particle constrained to move on an (N+1)-dim hypersphere.

3.1 The 3-dimensional hydrogen atom

The 3-dim hydrogen atom is known to possess the O(4) symmetry. We know that the irreps of O(4) are characterized by two labels μ_1, μ_2 and are denoted by (μ_1, μ_2), while the irreps of O(3) are characterized by one label $\mu_1' = L$ (the usual angular momentum quantum number) and are denoted by (μ_1'). When making the reduction O(4) $\supset$ O(3), μ_1' obtains all values permitted by the condition $\mu_1 \geq \mu_1' \geq \mu_2$ [26]. Furthermore, the decomposition O(3)$\supset$O(2) can be made, the irreps of O(2) characterized by the quantum number $M = L$, $L-1$, $L-2$, ..., $-(L-1)$, $-L$.

We are going to consider the completely symmetric irreps of O(4), which are of the form $(\mu_1, 0)$. The (LM) states contained in each O(4) irrep are shown in Table 5. The dimensions of the irreps are 1, 4, 9, 16, 25, ..., as expected from

Table 5

Decomposition of completely symmetric O(4) irreps (corresponding to the 3-dim hydrogen atom) using the O(4) ⊃ O(3) ⊃ O(2) chain. In the last column the states are labelled by (LM), where L is the O(3) quantum number and M is the O(2) one.

O(4)	O(3)	(LM)
(00)	(0)	(00)
(10)	(1) (0)	(11) (10) (1-1) (00)
(20)	(2) (1) (0)	(22) (21) (20) (2-1) (2-2) (11) (10) (1-1) (00)
(30)	(3) (2) (1) (0)	(33) (32) (31) (30) (3-1) (3-2) (3-3)
		(22) (21) (20) (2-1) (2-2) (11) (10) (1-1) (00)
(40)	(4) (3) (2) (1) (0)	(44) (43) (42) (41) (40) (4-1) (4-2) (4-3) (4-4)
		(33) (32) (31) (30) (3-1) (3-2) (3-3)
		(22) (21) (20) (2-1) (2-2) (11) (10) (1-1) (00)
(50)	(5) (4) (3) (2) (1) (0)	(55) (54) (53) (52) (51) (50) (5-1) (5-2) (5-3)
		(5-4) (5-5) (44) (43) (42) (41) (40) (4-1) (4-2)
		(4-3) (4-4) (33) (32) (31) (30) (3-1) (3-2) (3-3)
		(22) (21) (20) (2-1) (2-2) (11) (10) (1-1) (00)

eq. (8), since only the integer irreps occur. As pointed out by Ravenhall *et al.* [20], a bisection can be effected by inserting an impenetrable barrier through the center of the hydrogen atom. Only the states with $L-M$=odd remain then. ¿From Table 5 one sees that the remaining states are:

$\mu_1 = 1$: (10)

$\mu_1 = 2$: (10) (21) (2-1)

$\mu_1 = 3$: (10) (21) (2-1) (32) (30) (3-2)

$\mu_1 = 4$: (10) (21) (2-1) (32) (30) (3-2) (43) (41) (4-1) (4-3)

$\mu_1 = 5$: (10) (21) (2-1) (32) (30) (3-2) (43) (41) (4-1) (4-3) (54) (52) (50) (5-2) (5-4),

which correspond to degeneracies 1, 3, 6, 10, 15, ..., i.e. the degeneracies of U(3).

Keeping the states with $L-M$=even one is left with

$\mu_1 = 0$: (00)

$\mu_1 = 1$: (00) (11) (1-1)

$\mu_2 = 2$: (00) (11) (1-1) (22) (20) (2-2)

$\mu_1 = 3$: (00) (11) (1-1) (22) (20) (2-2) (33) (31) (3-1) (3-3)

$\mu_1 = 4$: (00) (11) (1-1) (22) (20) (2-2) (33) (31) (3-1) (3-3) (44) (42) (40) (4-2) (4-4)

$\mu_1 = 5$: (00) (11) (1-1) (22) (20) (2-2) (33) (31) (3-1) (3-3) (44) (42) (40) (4-2) (4-4) (55) (53) (51) (5-1) (5-3) (5-5).

The resulting degeneracies are again 1, 3, 6, 10, 15, 21, ..., i.e. U(3) degeneracies. Therefore a **bisection** of the 3-dim hydrogen atom, effected by choosing states with $\mathrm{mod}(L-M,2)=0$ or $\mathrm{mod}(L-M,2)=1$, is leading to two interleaving sets of U(3) states. Choosing states with $\mathrm{mod}(L+M,2)=0$ or 1 obviously leads to the same results.

The fact that by bisecting O(4) one obtains U(3) has been first pointed out by Ravenhall *et al.* [20]. Once the U(3) symmetry of the 3-dim HO is obtained, any further multisections on it will lead to RHO degeneracies, as pointed out in subsec. 2.1. We briefly show how this can be carried out by a few examples.

i) Selecting states with $\mathrm{mod}(L-M,4)=0$ gives

$\mu_1 = 0$: (00)

$\mu_1 = 1$: (00) (11)

$\mu_1 = 2$: (00) (11) (22) (2-2)

$\mu_1 = 3$: (00) (11) (22) (2-2) (33) (3-1)

$\mu_1 = 4$: (00) (11) (22) (2-2) (33) (3-1) (44) (40) (4-4),

i.e. it leads to degeneracies 1, 2, 4, 6, 9, ..., which are those of the 2:2:1 RHO. The same result is obtained by choosing states with $\mathrm{mod}(L-M,4)=2$. Therefore the operation of dividing the states with $\mathrm{mod}(L-M,2)=0$ of the 3-dim HA according to the $\mathrm{mod}(L-M,4)$ is equivalent to a bisection of the 3-dim HO. The same holds for $\mathrm{mod}(L+M,4)$, as well as for dividing the $\mathrm{mod}(L-M,2)=1$ states of the 3-dim HA according to $\mathrm{mod}(L-M,4)=1$ or 3.

ii) Selecting states with $\mathrm{mod}(L-M,6)=0$, or 2, or 4 (or 1, or 3, or 5) leads to degeneracies 1, 2, 3, 5, 7, 9, 12, ..., i.e. to the degeneracies of the 3:3:1 RHO. Therefore this operation is equivalent to a trisection of the 3-dim HO. The same holds for $\mathrm{mod}(L+M,6)$.

iii) Selecting states with $\mathrm{mod}(L-M,8)=0$, or 2, or 4, or 6 (or 1, or 3, or 5,

Table 6

Decomposition of completely symmetric O(5) irreps (corresponding to the 4-dim hydrogen atom) using the O(5) $\supset$ O(4) $\supset$ O(3) chain.

O(5)	O(4)	O(3)
(00)	(00)	(0)
(10)	(10) (00)	$(1)\ (0)^2$
(20)	(20) (10) (00)	$(2)\ (1)^2\ (0)^3$
(30)	(30) (20) (10) (00)	$(3)\ (2)^2\ (1)^3\ (0)^4$
(40)	(40) (30) (20) (10) (00)	$(4)\ (3)^2\ (2)^3\ (1)^4\ (0)^5$

or 7) leads to degeneracies 1, 2, 3, 4, 6, 8, 10, 12, ..., i.e. to the degeneracies of the 4:4:1 RHO. Therefore this operation is equivalent to a tetrasection of the 3-dim HO. The same holds for $\mathrm{mod}(L+M,8)$.

Combining two of the above operations one obtains the results corresponding to the appropriate multisections of the HO. Thus:

i) Selecting states with $\mathrm{mod}(L-M,4)=0$ and $\mathrm{mod}(L+M,4)=0$ one finds the degeneracies 1, 1, 3, 3, 6, 6, 10, 10, ..., which correspond to the 2:1:1 RHO.

ii) Selecting states with $\mathrm{mod}(L-M,6)=0$ and $\mathrm{mod}(L+M,6)=0$ one finds the degeneracies 1, 1, 1, 3, 3, 3, 6, 6, 6, ..., which characterize the 3:1:1 RHO.

iii) Selecting states with $\mathrm{mod}(L-M,8)=0$ and $\mathrm{mod}(L+M,8)=0$ one finds the degeneracies 1, 1, 1, 1, 3, 3, 3, 3, 6, 6, 6, 6, ..., which correspond to the 4:1:1 RHO.

3.2 The 4-dimensional hydrogen atom

The 4-dim hydrogen atom is characterized by the O(5) symmetry. The irreps of O(5) can be labelled as (μ_1,μ_2), while the irreps of O(4) can be labelled by (μ'_1,μ'_2). When making the reduction O(5)$\supset$O(4), μ'_1 and μ'_2 take the values permitted by the relation $\mu_1 \geq \mu'_1 \geq \mu_2 \geq \mu'_2$ [26]. Continueing further the reduction O(5)$\supset$O(4)$\supset$O(3)$\supset$O(2) one obtains the lists of states given it Table 6. The dimensions of the irreps are 1, 5, 14, 30, 55, ..., as expected from eq. (10), since only the integer irreps occur.

Selecting states with $\mathrm{mod}(L-M,2)=0$ or 1 one obtains the degeneracies 1, 4, 10, 20, 35, ..., which characterize U(4), i.e. the 4-dim isotropic HO 1:1:1:1.

Bisecting these results using $\mathrm{mod}(L-M,4)$ one obtains the degeneracies 1,

Table 7

Decomposition of completely symmetric O(6) irreps (corresponding to the 5-dim hydrogen atom) using the O(6) $\supset$ O(5) $\supset$ O(4) $\supset$ O(3) chain.

O(6)	O(5)	O(4)	O(3)
(000)	(00)	(00)	(0)
(100)	(10) (00)	(10) $(00)^2$	(1) $(0)^3$
(200)	(20) (10) (00)	(20) $(10)^2$ $(00)^3$	(2) $(1)^3$ $(0)^6$
(300)	(30) (20) (10) (00)	(30) $(20)^2$ $(10)^3$ $(00)^4$	(3) $(2)^3$ $(1)^6$ $(0)^{10}$

3, 7, 13, 22, ..., which correspond to the 2:2:2:1 RHO, while trisecting them according to $\text{mod}(L-M,6)$ one obtains the degeneracies 1, 3, 6, 11, 18, ..., which correspond to the 3:3:3:1 RHO, and tetrasecting them according to $\text{mod}(L-M,8)$ one obtains the degeneracies 1, 3, 6, 10, ..., which are the degeneracies of the 4:4:4:1 RHO.

Combining the bisections $\text{mod}(L-M,4)$ and $\text{mod}(L+M,4)$ one obtains the degeneracies 1, 2, 5, 8, ... of the 2:2:1:1 RHO, while combining of the trisections $\text{mod}(L-M,6)$ and $\text{mod}(L+M,6)$ leads to the 1, 2, 3, 6, 9, ... degeneracies of the 3:3:1:1 RHO.

The similarity between the 2:2:1:1 degeneracies and the O(5) degeneracies can now be understood as due to the fact that the 2:2:1:1 degeneracies are obtained from the O(5) ones using the appropriate series of bisections described above.

3.3 The 5-dimensional hydrogen atom

The 5-dim hydrogen atom is characterized by the O(6) symmetry, the irreps of which can be labelled as (μ_1,μ_2,μ_3), while the irreps of O(5) can be labelled as (μ_1',μ_2'). In the reduction O(6)$\supset$O(5) the labels μ_1' and μ_2' have to satisfy the conditions $\mu_1 \geq \mu_1' \geq \mu_2 \geq \mu_2' \geq \mu_3$ [26]. Continueing further the reduction O(6)$\supset$O(5)$\supset$O(4)$\supset$O(3)$\supset$O(2) one obtains the results of Table 7. The dimensions of the irreps are 1, 6, 20, 50, ..., as expected from eq. (12), since only integer irreps occur.

Selecting states with $\text{mod}(L-M,2)=0$ or 1 one obtains the degeneracies 1, 5, 15, 35, ... which characterize U(5), i.e. the isotropic 5-dim HO 1:1:1:1:1.

A further bisection using $\text{mod}(L-M,4)$ leads to the degeneracies of the 2:2:2:2:1 RHO, while an additional bisection of these results using $\text{mod}(L+M,4)$ leads to the degeneracies of the 2:2:2:1:1 RHO.

Table 8

Decomposition of completely symmetric O(7) irreps (corresponding to the 6-dim hydrogen atom) using the O(7) $\supset$ O(6) $\supset$ O(5) $\supset$ O(4) $\supset$ O(3) chain.

O(7)	O(6)	O(5)
(000)	(000)	(00)
(100)	(100) (000)	(10) $(00)^2$
(200)	(200) (100) (000)	(20) $(10)^2$ $(00)^3$
(300)	(300) (200) (100) (000)	(30) $(20)^2$ $(10)^3$ $(00)^4$

O(4)	O(3)
(00)	(0)
(10) $(00)^3$	(1) $(0)^4$
(20) $(10)^3$ $(00)^6$	(2) $(1)^4$ $(0)^{10}$
(30) $(20)^3$ $(10)^6$ $(00)^{10}$	(3) $(2)^4$ $(1)^{10}$ $(0)^{20}$

3.4 The 6-dimensional hydrogen atom

The 6-dim hydrogen atom is characterized by the O(7) symmetry, the irreps of which are labelled by (μ_1, μ_2, μ_3), while the irreps of O(6) can be labelled by (μ'_1, μ'_2, μ'_3). In the reduction O(7)$\supset$O(6) the labels μ'_1, μ'_2, μ'_3 have to satisfy the condition $\mu_1 \geq \mu'_1 \geq \mu_2 \geq \mu'_2 \geq \mu_3 \geq \mu'_3$ [26]. Further continueing the reduction O(7)$\supset$O(6)$\supset$O(5)$\supset$O(4)$\supset$O(3)$\supset$O(2) one obtains the results of Table 8. The dimensions of the irreps are 1, 7, 27, 77, ..., as expected from eq. (14), since only integer irreps occur.

Selecting states with $\mathrm{mod}(L-M, 2) = 0$ or 1 one obtains the degeneracies 1, 6, 21, 56, ... which characterize U(6), i.e. the isotropic 6-dim HO 1:1:1:1:1:1.

A further bisection using $\mathrm{mod}(L-M, 4)$ leads to the degeneracies of the 2:2:2:2:2:1 RHO, while an additional bisection of these results using $\mathrm{mod}(L+M, 4)$ leads to the degeneracies of the 2:2:2:2:1:1 RHO.

3.5 The N-dimensional hydrogen atom

The N-dim hydrogen atom is characterized by the O(N+1) symmetry. Only the completely symmetric irreps of O(N+1) occur. Using the chain O(N+1) $\supset$ O(N) $\supset \ldots \supset$ O(3) $\supset$ O(2) one can find the (LM) states contained in each O(N+1) irrep. Bisecting them using $\mathrm{mod}(L-M, 2) = 0$ or 1 one is left with the irreps of U(N). Further multisections of the U(N) irreps lead to the appropriate symmetries of the N-dim RHO. It is therefore clear that all symmetries of the N-dim RHO can be obtained from a common parent, the O(N+1) symmetry.

Thus it is not surprising that some of them (notably the 2:2:1:...:1 ones) show similarities to the corresponding O(N+1) symmetry. However, the only case in which an N-dim RHO symmetry is identical to an O(N+1) symmetry occurs for N=3, for which the 2:2:1 RHO symmetry is O(4) [20]. The rest of the RHO symmetries are not related to any orthogonal symmetries.

4 Discussion

The concept of bisection of an N-dim isotropic harmonic oscillator with U(N) symmetry, introduced by Ravenhall *et al.* [20], has been generalized. Trisections, tetrasections, ..., n-sections of the N-dim isotropic harmonic oscillator have been introduced. They are shown to lead to the various symmetries of the anisotropic N-dim harmonic oscillator with rational ratios of frequencies (RHO). Furthermore, multisections of the N-dim hydrogen atom with O(N+1) symmetry have been considered. It is shown that a bisection of O(N+1) leads to U(N), so that further multisections just lead to various cases of the N-dim RHO. The opposite does not hold, i.e. multisections of U(N) do not lead to O(N+1) symmetries, the only exception being the bisection of U(3) which does lead to O(4). Even in the case of the 4-dim HO, which has the U(4) symmetry, which is isomorphic to O(6) and has an O(5) subalgebra, no multisection, or combination of multisections, leading to a RHO with O(5) symmetry has been found. We conclude therefore that the rich variety of the N-dim RHO symmetries have a common "parent", the U(N) symmetry of the N-dim isotropic harmonic oscillator or the O(N+1) symmetry of the N-dim hydrogen atom, but they are not in general related to unitary or orthogonal symmetries themselves.

Since the RHO is of current interest in relation to various physical systems (superdeformed and hyperdeformed nuclei [8–12], Bloch–Brink α-cluster model [15–17], deformed atomic clusters [18,19]), the unification of the rich variety of symmetries appearing in the RHO for different frequency ratios in a common algebraic framework is an interesting project. In [6] the 3-dim RHO degeneracies are obtained as reducible representations of U(3). It could be possible to construct an algebraic framework in which the RHO degeneracies occur as irreducible representations of an appropriate algebra. Work in this direction is in progress [21].

Throughout this paper the properties of the completely symmetric irreps of U(N) and O(N+1) have been considered. Similar studies of completely antisymmetric irreps, or irreps with mixed symmetry, might be worth exploring.

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