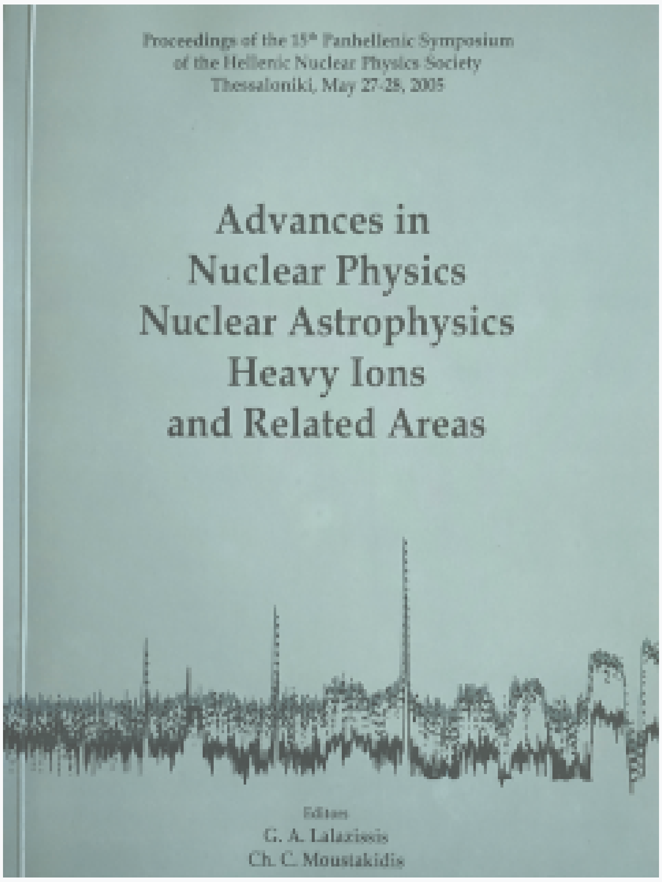
**HNPS Advances in Nuclear Physics**

Vol 14 (2005)

HNPS2005




**On hidden symmetries of the Skyrme model**

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doi: 10.12681/hnps.2269

# On hidden symmetries of the Skyrme model

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**Abstract**

We present a preliminary attempt to establish the existence of hidden nonlinear symmetries of the Skyrme model which could lead to the further integration of the system. An explicit illustration is given for the SU(2) symmetry group.

The Skyrme model [1] represents a nonlinear lagrangian which supports the existence of stable topological solitons. In terms of the Skyrme field $U(x) \in SU(N)$ it reads

$$\mathcal{L} = \frac{F_\pi^2}{16}\,\mathrm{Tr}\left(\partial_\mu U^\dagger\,\partial_\mu U\right) + \frac{1}{32e^2}\,\mathrm{Tr}\left(\left[U^\dagger\,\partial_\mu U, U^\dagger\,\partial_\nu U\right]^2\right), \qquad (1)$$

where $e$ is a constant and $F_\pi$ is the pion decay constant.

The matrix field $U(x)$ can be written in terms of the generators of $SU(N)$ $T_\alpha$ , with $\alpha = 1,\ldots,N^2-1$, in the following form $U(x) = e^{iL^\alpha T_\alpha}$. By taking the logarithm in this relation we get

$$-i\ln U(x) = L^\alpha T_\alpha. \qquad (2)$$

Then, by differentiating (2) with respect to $x^\mu$ we introduce the left–invariant Maurer–Cartan covariant vector

$$-i\, U^\dagger\, \partial_\mu U = L_\mu^\alpha\, T_\alpha \equiv \boldsymbol{L}_\mu, \tag{3}$$

which represents an $SU(N)$ Lie–algebra valued current. In terms of $\boldsymbol{L}_\mu(x)$, the Skyrme lagrangian density (1) reads

$$\mathcal{L} = \frac{F_\pi^2}{16}\, \mathrm{Tr}\,(\boldsymbol{L}_\mu\, \boldsymbol{L}_\mu) + \frac{1}{32e^2}\, \mathrm{Tr}\left([\boldsymbol{L}_\mu, \boldsymbol{L}_\nu]^2\right). \tag{4}$$

By making use of the standard variational principle, from the Skyrme model (4) one can derive the Euler-Lagrange's equation

$$\partial_\mu \left( F_\pi^2\, \boldsymbol{L}_\mu + \frac{1}{e^2}\, [\boldsymbol{L}_\nu, [\boldsymbol{L}_\mu, \boldsymbol{L}_\nu]] \right) = 0. \tag{5}$$

In [1] Skyrme pointed out that the $SU(2)$ static field configuration with the spherically symmetric ansatz

$$U(\vec{r}) = \cos\theta(r) + i\,\hat{r}_a \tau_a\, \sin\theta(r), \tag{6}$$

with unitary vector $\hat{r} = \vec{r}/r$, yields a nontrivial solution of the field equation (5) which realizes the absolute minimum of the energy in the first homotopy class, i.e., among fields with $|\boldsymbol{Q}| = 1$. By looking at the field equation (5), one realizes that the first term corresponds to a standard chiral field equation, whose properties and exact solutions are well studied. In particular, under the assumption of some symmetry conditions, this chiral equation can be completely integrable. The second term in (5) corresponds to the nonlinear Skyrme term. Thus, it is natural to think that if there exists some kind of relationship between the first and second terms of (5), the solutions corresponding to the former term could be mapped onto solutions of the latter, leading to the integration of the field equation (5). Of course, if such a symmetry does exist, it will be a nonlinear symmetry which involves derivatives of the functions that parameterize the covariant vector $\boldsymbol{L}_\mu(x)$ due to the nonlinear structure of (5). For simplicity we shall consider the simplest $SU(2)$ case. Thus, the Skyrme field has three generators (the Pauli matrices) $U(x) = e^{iL^\alpha T_\alpha}$, where $T_\alpha = \sigma_\alpha$, and allows the following parametrization

$$U(x) = \begin{pmatrix} e^{i\xi}\cos\eta & e^{i\theta}\sin\eta \\ -e^{-i\theta}\sin\eta & e^{-i\xi}\cos\eta \end{pmatrix}, \tag{7}$$

where the real functions $\xi(x)$, $\theta(x)$ and $\eta(x)$ depend on $x_\mu$. By computing $\boldsymbol{L}_\mu(x)$ we obtain the following expressions for $L_\mu^\alpha$:

$$\begin{aligned} L_\mu^1 &= \sin(\theta-\xi)\ \eta_\mu + \cos(\theta-\xi)\sin\eta\cos\eta\ (\theta+\xi)_\mu \\ L_\mu^2 &= \cos(\theta-\xi)\ \eta_\mu - \sin(\theta-\xi)\sin\eta\cos\eta\ (\theta+\xi)_\mu \\ L_\mu^3 &= \cos^2\eta\ \xi_\mu - \sin^2\eta\ \theta_\mu \end{aligned} \tag{8}$$

where $\eta_\mu \equiv \partial_\mu \eta$, etc. Here it is useful to denote the second term involved in the "conserved chiral current" of (5) as follows

$$\lambda\,[\boldsymbol{L}_\nu, [\boldsymbol{L}_\mu, \boldsymbol{L}_\nu]] \equiv \boldsymbol{P}_\mu = P_\mu^\alpha T_\alpha, \tag{9}$$

where we have introduced the effective constant $\lambda = 1/(F_\pi^2 e^2)$. By computing the components $P_\mu^\alpha$ of the vector $\boldsymbol{P}_\mu(x)$ we get

$$\begin{aligned} P_\mu^1 =& \sin(\theta-\xi)\ \left\{\cos^2\eta\ \xi_\nu\ [\xi_\mu, \eta_\nu] + \sin^2\eta\ \theta_\nu\ [\theta_\mu, \eta_\nu]\right\} \\ &+ \cos(\theta-\xi)\sin\eta\cos\eta\ \left\{\left(\cos^2\eta\ \xi_\nu - \sin^2\eta\ \theta_\nu\right)[\xi_\mu, \theta_\nu] \right.\\ &\left.+ \eta_\nu\,[\eta_\mu, (\theta+\xi)_\nu]\right\} \\ P_\mu^2 =& \cos(\theta-\xi)\ \left\{\cos^2\eta\ \xi_\nu\ [\xi_\mu, \eta_\nu] + \sin^2\eta\ \theta_\nu\ [\theta_\mu, \eta_\nu]\right\} \\ &+ \sin(\theta-\xi)\sin\eta\cos\eta\ \left\{\left(\cos^2\eta\ \xi_\nu - \sin^2\eta\ \theta_\nu\right)[\theta_\mu, \xi_\nu] \right.\\ &\left.+ \eta_\nu\,[(\theta+\xi)_\mu, \eta_\nu]\right\} \\ P_\mu^3 =& \sin^2\eta\cos^2\eta\ (\theta+\xi)_\rho\ [\theta_\mu, \xi_\rho] + \cos^2\eta\ \eta_\nu\ [\eta_\mu, \xi_\nu] \\ &+ \sin^2\eta\ \eta_\nu\ [\theta_\mu, \eta_\nu]\,. \end{aligned} \tag{10}$$

Here it will be useful to introduce the following notation

$$
\begin{aligned}
L^1_\mu &= \chi_\mu \qquad & P^1_\mu &= G_\mu \\
L^2_\mu &= \psi_\mu \qquad & P^2_\mu &= H_\mu \\
L^3_\mu &= \varphi_\mu \qquad & P^3_\mu &= F_\mu
\end{aligned}
\tag{11}
$$

where $\chi_\mu$, $\psi_\mu$ and $\varphi_\mu$ are defined through the relations (8) and

$$
\begin{aligned}
G_\mu = & \big\{-\cos(\theta-\xi)\sin\eta\cos\eta\ \left[\eta^2+\cos^2\eta\ \xi^2-\sin^2\eta\ (\theta\,\xi)\right] \\
& +\sin(\theta-\xi)\sin^2\eta\ (\theta\,\eta)\big\}\,\theta_\mu \\
& +\big\{-\cos(\theta-\xi)\sin\eta\cos\eta\ \left[\eta^2-\cos^2\eta\ (\theta\,\xi)+\sin^2\eta\ \theta^2\right] \\
& +\sin(\theta-\xi)\cos^2\eta\ (\xi\,\eta)\big\}\,\xi_\mu \\
& +\{\cos(\theta-\xi)\sin\eta\cos\eta\ \left[(\theta\,\eta)+(\xi\,\eta)\right] \\
& -\sin(\theta-\xi)\left[\cos^2\eta\ \xi^2+\sin^2\eta\ \theta^2\right]\big\}\,\eta_\mu \\
H_\mu = & \big\{-\sin(\theta-\xi)\sin\eta\cos\eta\ \left[\eta^2+\cos^2\eta\ \xi^2-\sin^2\eta\ (\theta\,\xi)\right] \\
& +\cos(\theta-\xi)\sin^2\eta\ (\theta\,\eta)\big\}\,\theta_\mu \\
& +\big\{\sin(\theta-\xi)\sin\eta\cos\eta\ \left[\eta^2-\cos^2\eta\ (\theta\,\xi)+\sin^2\eta\ \theta^2\right] \\
& +\cos(\theta-\xi)\cos^2\eta\ (\xi\,\eta)\big\}\,\xi_\mu \\
& -\big\{\sin(\theta-\xi)\sin\eta\cos\eta\ \left[(\theta\,\eta)+(\xi\,\eta)\right] \\
& +\cos(\theta-\xi)\left[\cos^2\eta\ \xi^2+\sin^2\eta\ \theta^2\right]\big\}\,\eta_\mu \\
F_\mu = & \big\{\sin^2\eta\cos^2\eta\ \left[(\theta\,\xi)+\xi^2\ \right]+\sin^2\eta\ \eta^2\big\}\,\theta_\mu \\
& -\big\{\sin^2\eta\cos^2\eta\ \left[(\theta\,\xi)+\theta^2\ \right]+\cos^2\eta\ \eta^2\big\}\,\xi_\mu \\
& +\left[\cos^2\eta\ (\xi\,\eta)-\sin^2\eta\ (\theta\,\eta)\right]\eta_\mu .
\end{aligned}
\tag{12}
$$

where, for instance, $\xi^2=(\xi_\nu\cdot\xi_\nu)$, $(\xi\,\eta)=(\xi_\nu\cdot\eta_\nu)$, etc.

We are looking for a symmetry between the vectors $\boldsymbol{L}_\mu$ and $\boldsymbol{P}_\mu$

$$
\boldsymbol{L}_\mu(x) \qquad \longleftrightarrow \qquad \boldsymbol{P}_\mu(x), \tag{13}
$$

where now

$$\boldsymbol{L}_\mu(x) = \begin{pmatrix} \xi_\mu & \varphi_\mu - i\psi_\mu \\ \varphi_\mu + i\psi_\mu & -\xi_\mu \end{pmatrix} \tag{14}$$

and

$$\boldsymbol{P}_\mu(x) = \lambda \begin{pmatrix} F_\mu & G_\mu - iH_\mu \\ G_\mu + iH_\mu & -F_\mu \end{pmatrix}. \tag{15}$$

In the particular case in which $\theta(x) = \xi(x)$, the components of the chiral vectors $\boldsymbol{L}_\mu$ and $\boldsymbol{P}_\mu$ read

$$\begin{aligned} \varphi_\mu &= \sin(2\eta)\ \xi_\mu & \longleftrightarrow \quad & G_\mu = \lambda \sin(2\eta) \left[(\xi\,\eta)\eta_\mu - \eta^2\ \xi_\mu\right], \\ \psi_\mu &= \eta_\mu & \longleftrightarrow \quad & H_\mu = \lambda \left[(\xi\,\eta)\xi_\mu - \xi^2\ \eta_\mu\right], \\ \chi_\mu &= \cos(2\eta)\ \xi_\mu & \longleftrightarrow \quad & F_\mu = \lambda \cos(2\eta) \left[(\xi\,\eta)\eta_\mu - \eta^2\ \xi_\mu\right], \end{aligned} \tag{16}$$

By choosing the following initial functions for the vector $\boldsymbol{L}_\mu$

$$\varphi_{0\mu} = \frac{\sin(2\eta)\left[(\xi\,\eta)\eta_\mu - \eta^2\ \xi_\mu\right]}{\sqrt[3]{\lambda\left[\xi^2\ \eta^2 - (\xi\,\eta)^2\right]^2}} \tag{17}$$

$$\psi_{0\mu} = \frac{(\xi\,\eta)\xi_\mu - \xi^2\ \eta_\mu}{\sqrt[3]{\lambda\left[\xi^2\ \eta^2 - (\xi\,\eta)^2\right]^2}} \tag{18}$$

$$\chi_{0\mu} = \frac{\cos(2\eta)\left[(\xi\,\eta)\eta_\mu - \eta^2\ \xi_\mu\right]}{\sqrt[3]{\lambda\left[\xi^2\ \eta^2 - (\xi\,\eta)^2\right]^2}}, \tag{19}$$

and computing the components of the current $\boldsymbol{P}_\mu$, we obtain

$$\begin{aligned} G_\mu &= \sin(2\eta)\ \xi_\mu \equiv \varphi_\mu \\ H_\mu &= \eta_\mu \equiv \psi_\mu \\ F_\mu &= \cos(2\eta)\ \xi_\mu \equiv \chi_\mu \end{aligned} \tag{20}$$

which precisely coincide with the components of the vector $\boldsymbol{L}_\mu$. Thus, the initial functions (17)–(19) map the components of the chiral current $\boldsymbol{P}_\mu$ onto the components of the chiral vector $\boldsymbol{L}_\mu$, establishing a nontrivial symmetry between them.

Obviously, the generalization of this relationship to the general case in which the restriction $\theta(x) = \xi(x)$ does not hold, constitutes a nontrivial task. It is interesting to try to apply the obtained nonlinear symmetry in order to generate exact solutions to the field equation (5) of the Skyrme model. Even if this constitutes a rather difficult task in the general case, it could be analytically treatable for some simple nontrivial initial solutions. It is worth noticing that a first attempt to obtain the spherically symmetric ansatz (6) or axially symmetric field configurations compatible with the special condition $\theta(x) = \xi(x)$ (or some equivalent restrictions) was unsuccessful. It seems that it is necessary to consider the three nontrivial different functions $\xi(x)$, $\theta(x)$ and $\eta(x)$ in order to achieve this aim. This topics are currently under investigation.

**Acknowledgments.** AHA and KK are grateful to V. Oikonomou for useful discussions while this investigation was carried out. AHA acknowledges a grant for postdoctoral studies provided by the Greek Government. This research was supported by grants CIC-UMSNH-4.16 and CONACYT-F42064.

## References

[1] T. H. R. Skyrme, Nucl. Phys. **31**, 556 (1961). See also A.P. Balachandran, "The Skyrme soliton: a review", in *Solitons in nuclear and elementary particle physics*, Ed. A. Chodos, E. Hadjimichael and C Tze (World Scientific, 1984) and V.G. Makhan'kov, Yu.P. Rybakov and V.I. Sanuyk, *The Skyrme model: Fundamentals, methods, applications,* (Berlin, Springer, 1993); Sov. Phys. Usp. **35**, 55 (1992), Usp. Fiz. Nauk **162**, 1 (1992).