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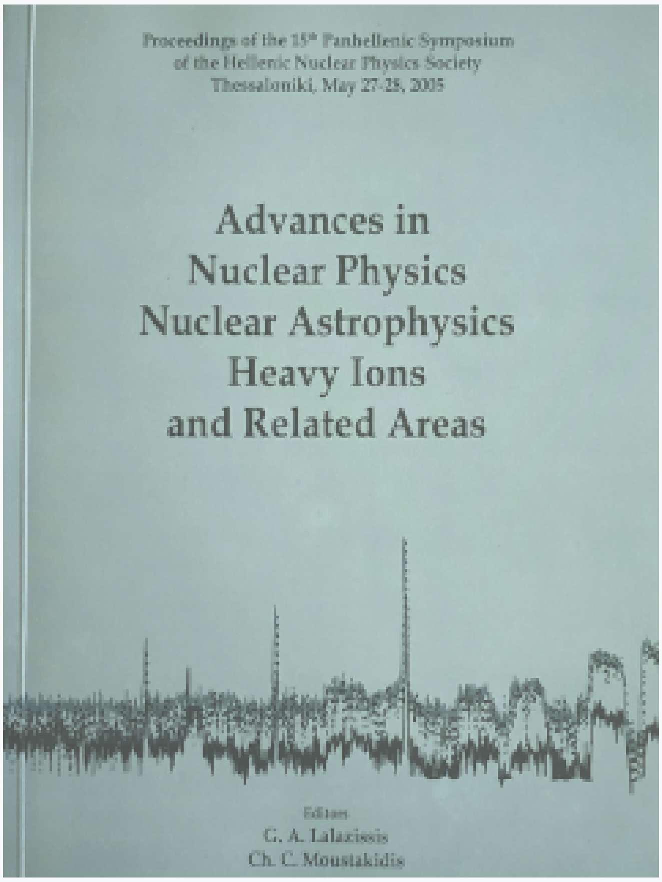

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γ-rigid solution of the Bohr Hamiltonian compared to the E(5) critical point symmetry

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Abstract

A γ-rigid solution of the Bohr Hamiltonian for $\gamma = 30^\circ$ is derived, its ground state band being related to the second order Casimir operator of the Euclidean algebra E(4). Parameter-free (up to overall scale factors) predictions for spectra and B(E2) transition rates are in close agreement to the E(5) critical point symmetry, as well as to experimental data in the Xe region around $A = 130$.

1 Introduction

The E(5) critical point symmetry [1] has been obtained as an exact solution of the Bohr Hamiltonian [2] for γ-independent potentials, while the X(5) model is obtained as an approximate solution for $\gamma \approx 0^\circ$ [3]. Another approximate solution, with $\gamma \approx 30^\circ$, called Z(5), has also been obtained [4]. In all these cases, five degrees of freedom (the collective variables β, γ, and the three Euler angles) are taken into account.

In the present work we derive an exact solution of the Bohr Hamiltonian for $\gamma = 30^\circ$, by "freezing" γ (as in Ref. [5]) to this value and taking into account only four degrees of freedom (β and the Euler angles). In accordance to previous terminology, this solution will be called Z(4).

The Z(4) solution will be introduced in Section 2 and its ground state band will be related to the Euclidean algebra E(4) in Section 3. Numerical results and comparisons to E(5) and experiment will be given in Section 4, while discussion of the present results and plans for further work will appear in Section 5.

2 The Z(4) model

In the model of Davydov and Chaban [5] it is assumed that the nucleus is rigid with respect to γ-vibrations. Then the Hamiltonian depends on four variables (β, θ_i) and has the form [5]

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^3}\frac{\partial}{\partial\beta}\beta^3\frac{\partial}{\partial\beta} - \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2(\gamma - \frac{2\pi}{3}k)}\right] + U(\beta), \tag{1}$$

where β and γ are the usual collective coordinates [2], while Q_k $(k = 1, 2, 3)$ are the components of angular momentum and B is the mass parameter.

Introducing [1] reduced energies $\epsilon = (2B/\hbar^2)E$ and reduced potentials $u = (2B/\hbar^2)U$, and considering a wave function of the form $\Psi(\beta, \theta_i) = \phi(\beta)\psi(\theta_i)$, where θ_i ($i = 1, 2, 3$) are the Euler angles, separation of variables leads to two equations

$$\left[\frac{1}{\beta^3}\frac{\partial}{\partial\beta}\beta^3\frac{\partial}{\partial\beta} - \frac{\lambda}{\beta^2} + (\epsilon - u(\beta))\right]\phi(\beta) = 0, \tag{2}$$

$$\left[\frac{1}{4}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2(\gamma - \frac{2\pi}{3}k)} - \lambda\right]\psi(\theta_i) = 0. \tag{3}$$

In the case of $\gamma = \pi/6$, the last equation has been solved by Meyer-ter-Vehn [6], with $\lambda = \lambda_{L,\alpha} = L(L+1) - 3\alpha^2/4$, where α are the eigenvalues of the projection of angular momentum on the body-fixed $\hat{x}'$-axis. α has to be an even integer [6].

Instead of the projection α of the angular momentum on the $\hat{x}'$-axis, it is customary to introduce the wobbling quantum number [6] $n_w = L - \alpha$, which labels a series of bands with $L = n_w, n_w + 2, n_w + 4, \ldots$ (with $n_w > 0$) next to the ground state band (with $n_w = 0$) [6].

The "radial" Eq. (2) is exactly soluble in the case of an infinite square well potential ($u(\beta) = 0$ for $\beta \leq \beta_W$, $u(\beta) = \infty$ for $\beta > \beta_W$). Using the transformation $\phi(\beta) = \beta^{-1}f(\beta)$, Eq. (2) becomes a Bessel equation

$$\left[\frac{\partial^2}{\partial\beta^2} + \frac{1}{\beta}\frac{\partial}{\partial\beta} + \left(\epsilon - \frac{\nu^2}{\beta^2}\right)\right]f(\beta) = 0, \quad \nu = \sqrt{\lambda + 1}. \tag{4}$$

Then the boundary condition $f(\beta_W) = 0$ determines the spectrum, $\epsilon_{\beta;s,\nu} = \epsilon_{\beta;s,n_w,L} = (k_{s,\nu})^2$, $k_{s,\nu} = x_{s,\nu}/\beta_W$, where $x_{s,\nu}$ is the sth zero of the Bessel function $J_\nu(k_{s,\nu}\beta)$. The ground state band corresponds to $s = 1$, $n_w = 0$. This model will be called the Z(4) model.

The calculation of B(E2)s proceeds as in Ref. [4], the only difference being that the volume element in the integrals over β contains β^3 instead of β^4, since it corresponds to four dimensions instead of five.

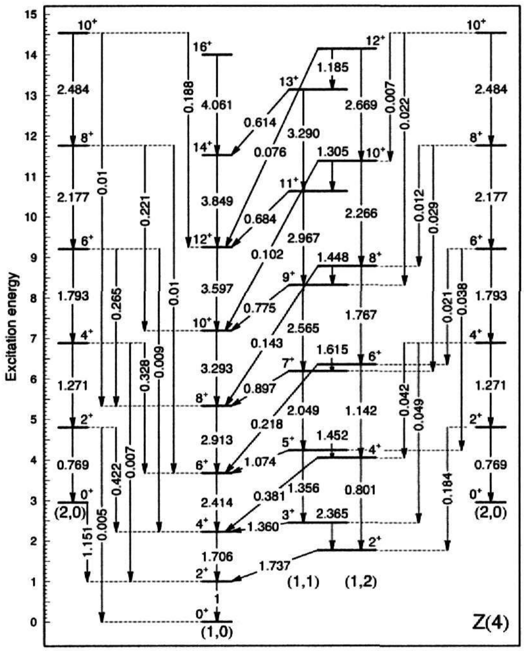

Fig. 1. Spectrum and intraband and interband B(E2) transition rates in the Z(4) model, normalized to the B(E2;$2_{1,0} \to 0_{1,0}$) rate. Bands are labelled by (s, n_w), their levels being normalized to $2_{1,0}$. The (2,0) band is shown both at the left and at the right end of the figure for drawing purposes.

3 Relation of the ground state band of Z(4) to E(4)

The ground state band of the Z(4) model is related to the second order Casimir operator of E(4), the Euclidean group in four dimensions. In order to see this, one can consider in general the Euclidean algebra in n dimensions, E(n), which is the semidirect sum of the algebra T_n of translations in n dimensions, generated by the momenta, and the SO(n) algebra of rotations in n dimensions, generated by the angular momenta, symbolically written as $\mathrm{E(n)} = \mathrm{T_n} \oplus_s \mathrm{SO(n)}$ [7]. One can see that the square of the total momentum, P^2, is a second order Casimir operator of the algebra, while the eigenfunctions of this operator satisfy the equation

$$\left(-\frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r}+\frac{\omega(\omega+n-2)}{r^2}\right)F(r)=k^2F(r), \tag{5}$$

in the left hand side of which the eigenvalues of the Casimir operator of SO(n), $\omega(\omega+n-2)$ appear. Putting $F(r)=r^{(2-n)/2}f(r)$, and $\nu=\omega+(n-2)/2$, Eq. (5) is brought into the form

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+k^2-\frac{\nu^2}{r^2}\right)f(r)=0, \tag{6}$$

the eigenfunctions of which are the Bessel functions $f(r)=J_\nu(kr)$. The similarity between Eqs. (6) and (4) is clear.

The ground state band of Z(4) is characterized by $n_w=0$, which means that $\alpha=L$. Then in Z(4) one obtains $\nu=L/2+1$, while in the case of E(4) one has $\nu=\omega+1$. The two results coincide for $L=2\omega$, i.e. for even values of L. One can easily see that this coincidence occurs only in four dimensions.

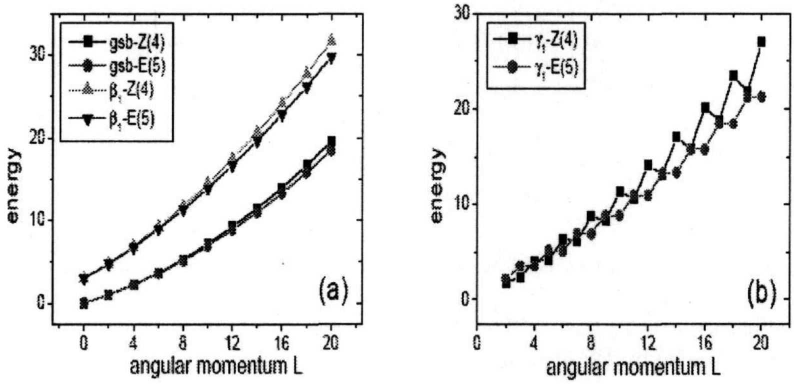

Fig. 2. (a) Ground state band $[(s,n_w)=(1,0)]$ and first excited band $[(s,n_w)=(2,0)$, labeled as β_1-band$]$ of Z(4) compared to the corresponding bands of E(5) [1]. In each model all levels are normalized to the 2_1^+ state. (b) The lowest "$K=2$ band" of Z(4) [formed out of the (s,n_w) bands (1,2) and (1,1), labeled as γ_1], compared to the corresponding band of E(5).

4 Numerical results and comparisons to E(5) and experiment

The level scheme of Z(4) is shown in Fig. 1. The similarity between the spectra of Z(4) and E(5) can be seen in Fig. 2(a), where the spectra of the ground state band and the β_1 band are given. One can easily check that the similarity extends to intraband and interband B(E2)s, for which the selection rules in the two models are the same.

The main difference between Z(4) and E(5) appears, as expected, in the γ_1 band, the spectrum of which is shown in Fig. 2(b). The predictions of the two models for the odd levels practically coincide, while the predictions for the even levels differ, since in the E(5) model the levels are exactly paired as (3,4), (5,6), (7,8), ..., as imposed by the underlying SO(5)⊃SO(3) symmetry [1], while in the Z(4) model the levels are approximately paired as (4,5), (6,7), (8,9), ..., which is a hallmark of rigid triaxial models [8]. The latter behaviour is never materialized fully, but it is known [8] that γ-unstable models and γ-rigid models yield similar predictions for most observables if γ_{rms} of the former equals γ_{rigid} of the latter, a situation occuring in the Ru-Pd, Xe-Ba (below $N = 82$), and Os-Pt regions.

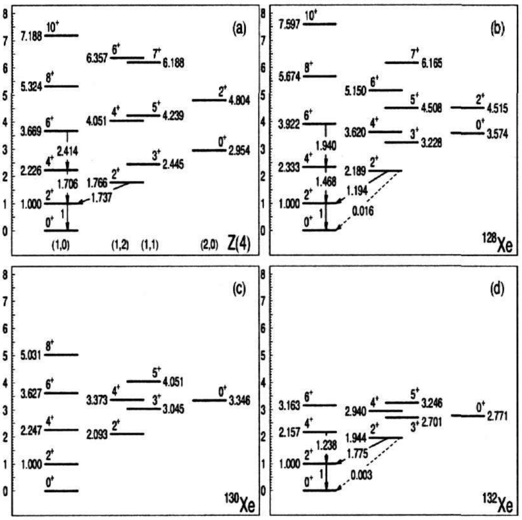

Fig. 3. Comparison of the Z(4) predictions for (normalized) energy levels and (normalized) B(E2) transition rates (a) to experimental data for ^{128}Xe [9] (b), ^{130}Xe [10] (c), and ^{132}Xe [11] (d). Bands in (a) are labelled by (s, n_w). See Section 4 for further discussion.

Predictions of the Z(4) model are compared to existing experimental data for ^{128}Xe [9], ^{130}Xe [10], and ^{132}Xe [11] in Fig. 3. The reasonable agreement observed is in no contradiction with the characterization of these nuclei as O(6) nuclei [8], since, as mentioned above, the predictions of γ-unstable models [like O(6)] and γ-rigid models [like Z(4)] for most observables are similar if γ_{rms} of the former equals γ_{rigid} of the latter.

5 Discussion

It should be emphasized that neither the similarity of spectra and B(E2) values of Z(4) to these of the E(5) model, nor the coincidence of the ground state band of Z(4) to the spectrum of the Casimir operator of the Euclidean algebra E(4) clarify the algebraic structure of the Z(4) model, the symmetry algebra of which has to be constructed explicitly, starting from the fact that the Bohr Hamiltonian for $\gamma = 30^{\circ}$ possesses "accidentally" a symmetry axis [6].

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