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# Parafermionic mapping of the two-colour delta model of interacting quarks

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## Abstract

Parafermions are used for creating an exact mapping of the two-colour delta model of interacting quarks, used in nuclear physics.

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## 1 Introduction

The derivation of the nuclear properties starting from the quark structure of the nucleons is an open problem. In a widely used approach, the quarks inside the nucleus form clusters, the nucleons, in a prescribed manner, which should ideally be explained by the strong interaction in the framework of QCD. In this kind of clustering, triplets of quarks are formed. This effect bears certain similarities with the pairing of nucleons in a closed nuclear shell [1,2], in which the clustering is associated with the formation of pairs of nucleons. In the quark case triplets of fermions are formed, while in the case of pairing pairs of fermions are involved in the interaction. Although the properties of the interactions in these two cases are very different, the effect of the clusterization reveals certain similarities, which allow similar techniques to be used in both cases.

One of the models created in the attempt to understand clusterization of quarks, is the model of Koltun *et al.* [3,4,5], in which the quarks move in one dimension and interact through a residual attractive delta-function potential. In the general case the quarks possess three colours, but a simplified version in which the quarks can only have two colours has also been developed. It is hoped that these models can give considerable insight into the way in which quarks condensate into nucleons. The two-colour form of the model

has an additional advantage: it reveals several mathematical similarities with the pairing of nucleons in a nuclear shell [1,2], allowing in this way techniques developed for the latter case to be applied in the former.

In the description of pairing in a single- $j$  nuclear shell model, correlated fermion pairs are used (see [6] and references therein), which satisfy commutation relations resembling boson commutation relations, including in addition corrections due to the presence of the Pauli principle. This fact has been the cause for the development of boson mapping techniques (see the reviews by Klein and Marshalek [7] and Hecht [8] and references therein), by which the description of systems of fermions in terms of bosons is achieved. Some of the boson mappings of the pairs of fermions are approximate, some others are exact but involve relatively complicated square roots of operators in the relevant formulas, these difficulties being caused by the presence of the Pauli principle. The boson mapping approach has also been applied in the case of the above mentioned quark models with two [9,10] or three [11,12] colours.

A simplified approach to these problems can be based on techniques making use of quantum algebras (quantum groups) [13]. For the case of pairing in a single- $j$  nuclear shell it has been found that it is possible to construct a generalized deformed oscillator in such a way, that the spectrum of the oscillator will exactly correspond to the pairing energy in the single- $j$  shell model, while the commutation relations of the relevant deformed bosons will exactly correspond to the commutation relations of the correlated fermion pairs in the single- $j$  shell under discussion [14]. It has also been seen that the generalized deformed bosons used in this case correspond to parafermions or deformed parafermions [15].

In this paper we attempt to get exact mappings for the two-colour delta model of interacting quarks mentioned above [3,4,5,9,10]. It turns out that an exact mapping of this model can be constructed through the use of parafermions.

In Section 2 of the present paper the algebraic structure of the two-colour delta model of interacting quarks will be clarified, while in Section 3 a parafermionic mapping of this model will be constructed. Finally Section 4 will contain comments on the present results and plans for further work.

## 2 Algebraic structure of the two-colour delta model

In the two-colour delta model [3,4,5,9,10] one considers a system of  $N$  non-relativistic quarks with colour  $c$ , which takes the two possible values  $\pm 1/2$ . The quarks move in a one-dimensional box of size  $L$  and interact through an attractive delta-function interaction.

Let  $q_{kc}^\dagger$  and  $q_{kc}$  be the creation and annihilation operators for a quark with momentum  $k$  and colour  $c$ . The Hamiltonian of this toy model is assumed to be [9]

$$H = \sum_{kc} \epsilon_k q_{kc}^\dagger q_{kc} - \frac{G}{2} \sum_{ijkl,c} q_{ic}^\dagger q_{j-c}^\dagger q_{l-c} q_{kc} \delta_{i+j,k+l}, \quad (1)$$

where  $\epsilon_k = k^2/2m$  and  $m$  is the mass of the quark.

The colourless “nucleons” are described by the operators [10]

$$\Gamma_{ij}^\dagger = \sum_c (-1)^{\frac{1}{2}-c} q_{ic}^\dagger q_{j-c}^\dagger. \quad (2)$$

After introducing the operators

$$N_{ij} = \sum_c q_{ic}^\dagger q_{jc}, \quad (3)$$

the Hamiltonian of Eq. (1) can be written in the following form:

$$H = \sum_k \epsilon_k N_{kk} - \frac{G}{4} \sum_{ijkl} \Gamma_{ij}^\dagger \Gamma_{kl} \delta_{i+j,k+l}. \quad (4)$$

The operators  $\Gamma_{ij}^\dagger$ ,  $\Gamma_{ij}$ , and  $N_{ij}$  close a  $\text{sp}(2N)$  Lie algebra,

$$\begin{aligned} [N_{ij}, \Gamma_{kl}^\dagger] &= \delta_{jl} \Gamma_{ik}^\dagger + \delta_{jk} \Gamma_{il}^\dagger, \\ [\Gamma_{kl}^\dagger, \Gamma_{ij}] &= \delta_{jl} N_{ki} + \delta_{jk} N_{li} + \delta_{ik} N_{lj} + \delta_{il} N_{kj}, \\ [N_{ij}, N_{kl}] &= \delta_{jk} N_{il} - \delta_{il} N_{jk}, \end{aligned} \quad (5)$$

obeying to the constraint relations

$$\begin{aligned} \text{for } i \neq j \quad (\Gamma_{ij}^\dagger)^3 &= 0, \\ \text{for } i = j \quad (\Gamma_{ii}^\dagger)^2 &= 0, \\ \text{for } i \neq j \quad (N_{ij})^3 &= 0, \\ \text{for } i = j \quad (N_{ii})^3 &= 3(N_{ii})^2 - 2N_{ii}. \end{aligned} \quad (6)$$

### 3 Parafermionic mapping

As it is well known, the non-constrained  $\text{sp}(2N)$  algebra can in general be realized exactly through a para-bosonic mapping using  $N$  parabosons of order  $p$  (see [16] and references therein). This mapping is related to the infinite dimensional representations of the  $\text{sp}(2N)$  algebra.

In the present special case, the constraints of Eq. (6) impose an exact mapping using  $N$  parafermions of order  $p = 2$ . This mapping is extremely simple

$$\begin{aligned}\Gamma_{ij}^\dagger &= \frac{1}{2} \{a_i^\dagger, a_j^\dagger\} = \frac{1}{2} (a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger), \\ N_{ij} &= \frac{1}{2} ([a_i^\dagger, a_j] + 2\delta_{ij}).\end{aligned}\tag{7}$$

The  $p$ -order parafermions  $a_i^\dagger, a_i$ , where  $i = 1, 2, \dots, N$ , satisfy the trilinear commutation relations [17,18]

$$\begin{aligned}[a_k, [a_\ell^\dagger, a_m]] &= 2\delta_{k\ell} a_m, \\ [a_k, [a_\ell^\dagger, a_m^\dagger]] &= 2\delta_{k\ell} a_m^\dagger - 2\delta_{km} a_\ell^\dagger, \\ [a_k, [a_\ell, a_m]] &= 0,\end{aligned}\tag{8}$$

together with the nilpotent conditions [17,18]

$$(a_i)^{p+1} = (a_j^\dagger)^{p+1} = 0.\tag{9}$$

The number operators

$$N_{ij} = \frac{1}{2} ([a_i^\dagger, a_j] + p\delta_{ij})\tag{10}$$

satisfy the commutation relations

$$[N_{ij}, a_k^\dagger] = \delta_{jk} a_i^\dagger, \quad [N_{ij}, a_k] = -\delta_{ik} a_j,\tag{11}$$

as well as the relation

$$N_{ii} (N_{ii} - 1) (N_{ii} - 2) \dots (N_{ii} - p) = 0.\tag{12}$$

After a little algebra we can show that the exact mapping of Eq. (7) on the parafermions of order  $p = 2$  satisfies the  $\text{sp}(2N)$  algebra of Eq. (5) obeying to the constraints of Eq. (6).

## 4 Discussion

We have shown that in the case of the two-colour delta model of interacting quarks [3,4,5,9,10] there is an exact mapping of quark pairs onto colourless parafermions of order  $p = 2$ . The situation resembles earlier findings for the single- $j$  nuclear shell, in which nucleon pairs are mapped onto parafermions of order related to the size of the nuclear shell [14].

The extreme simplicity of the present mapping has its roots in the trilinear and nilpotent relations [17,18] satisfied by the parafermions used here.

The extension of the present results to the case of the three-colour delta model of interacting quarks [3,4,5,11,12] is an obvious challenge. It is also of interest the extension of the present method to the study of the pairing problem in a multi- $j$  shell configuration.

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