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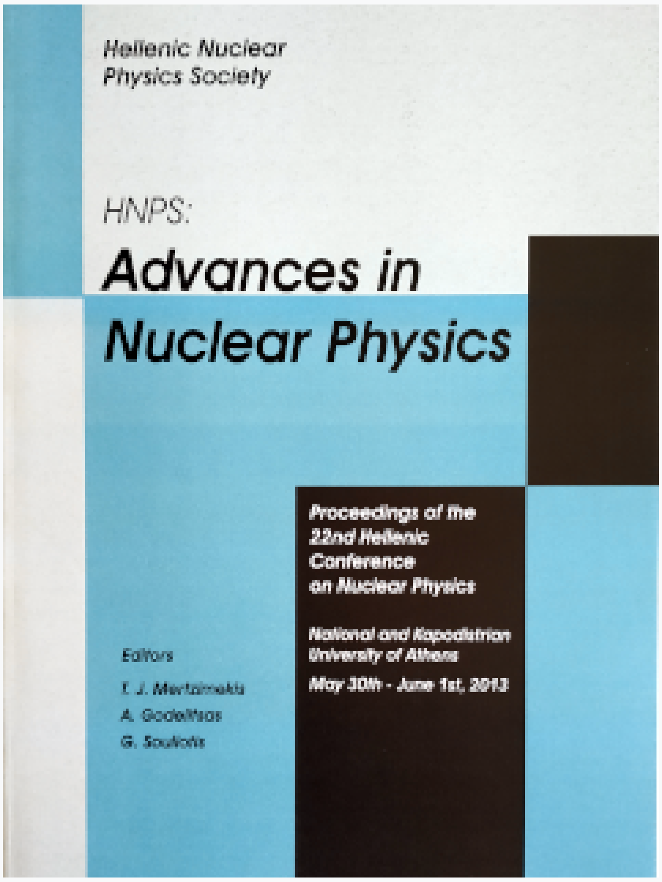

Phenomenological implications of Bohr space in six dimensions

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Abstract

The critical point for a second order shape/phase transition in the structural evolution of atomic nuclei, the consequences on the mass parameter and its irrotational flow are discussed after the embedding of Bohr space in six dimensions.

The phenomenology of the collective effects of nuclear spectra has traditionally been the range of application of the nuclear collective models, the most notable ones being the Interacting Boson Model [1] and the Bohr model [2]. During the last decade the interest in the exact solutions of the Bohr Hamiltonian was re-activated [3], mainly because of the proposal of the Critical Point Symmetries [5, 6] by Iachello. These are, $E(5)$ ($X(5)$), dynamical symmetries proposed to characterize the critical point for a second (first) order shape/phase transition in the structural evolution of atomic nuclei [7].

The fundamental theory of critical phenomena comes from Conformal Field Theories [8]. Characterization of the critical point by a dynamical symmetry, for a 2nd order phase transition like the $E(5)$, may lead to new manifestations of critical phenomena and their interpretation through Conformal Field Theories in the low energy regime of nuclear structure.

In the IBM nuclear collective motion is interpreted in terms of Interacting Bosons, paired valence nucleons of angular momentum zero (s boson) and two (**d** boson) [1], which build the $U(6)$ symmetry group. Three different chains of $U(6)$ subgroups define the dynamical symmetries of the Interacting Boson Model the $U(5)$, $SU(3)$, and $O(6)$. In its classical limit, obtained through the coherent states of $U(6)$ [9], the number of bosons extends to infinity. Dynamical symmetries are thus translated into phases of nuclear structure which accommodate spherical, axially symmetric, and γ-unstable shapes, through energy surfaces expressed in Bohr coordinates.

Shape/Phase transitions in the nuclear structure emerged with the classical limit of the IBM [9]. Initially named as ground state phase transitions [10] and first applied in the IBM [9], today are the Quantum Phase Transitions with intense applications in condensed matter systems [11]. The appropriate IBM Hamiltonian for the study of the shape/phase transitions is [12, 13]

$$H(\zeta, \chi) = c\left[(1-\zeta)\hat{n}_d - \frac{\zeta}{4N}\hat{Q}^\chi \cdot \hat{Q}^\chi\right], \tag{1}$$

where $\hat{n}_d = d^\dagger \cdot \tilde{d}$ is the number operator of the d bosons, $\hat{Q}^\chi = (s^\dagger \tilde{d} + d^\dagger s) + \chi(d^\dagger \tilde{d})^{(2)}$ is the Quadrupole operator, N is the total boson number and c is a scale factor. The above Hamiltonian contains the parameters ζ and χ, with $\zeta \in [0, 1]$ and $\chi \in [0, -\sqrt{7}/2 = -1.32]$. In this parameterization the $U(5)$ limit is obtained for $\zeta = 0$, the $O(6)$ limit for $\zeta = 1$, $\chi = 0$ and the $SU(3)$ limit for $\zeta = 1$, $\chi = -\sqrt{7}/2$. Gibbs criterion [14] for the existence of the critical point is satisfied by the large N limit in the classical limit, then for certain values of the control parameters ζ and χ the energy sufaces exhibit non-analyticities. According to the Ehrenfest classification [14] between the dynamical symmetries of the $U(5)$ and the $SU(3)$ there is a 1st order phase transition, and between the dynamical symmetries of $U(5)$ and $O(6)$ there is a 2nd order phase transition [1].

In the Bohr model, shapes of finite atomic nuclei, axially symmetric, triaxial, and close to spherical (γ-unstable), can be represented by specific constraints in the shape variables. The shape variables are the coordinates of the Bohr space [15] which parameterizes the quadrupole degree of freedom. It is defined by the element of length

$$ds^2 = g_{ij} dq^i dq^j, \tag{2}$$

with $q^1 = \beta$, $q^2 = \gamma$, $q^3 = \Theta$, $q^4 = \Phi$, $q^5 = \Psi$. They span a five dimensional space R^5, which is [16] the tensor product of the radial line R_+, representing the totality of the β values, and the unit four sphere S_4, representing the totality of the values of the angle γ and the three Euler angles Θ, Φ, ψ. Namely $R^5 \sim R_+ \times S_4$. The notation (β, Ω_4) for the Bohr coordinates q^i is useful, with Ω_4 representing the four angles. If $d\Omega_4$ is a line element on the unit four sphere S_4, then a consistent form of an R^5 line element with the definition of $R^5 \sim R_+ \times S_4$ is

$$ds^2 = d\beta^2 + \beta^2 d\Omega_4^2. \tag{3}$$

The line element is enough for the production of the Bohr Hamiltonian H_B. It is created by a mass parameter B and the Laplace-Beltrami operator $\nabla^2 = \sum_{ij} \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j$ of (3) which gives

$$H_B = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\Lambda^2\right], \tag{4}$$

with Λ^2 the second order Casimir operator of $SO(5)$ which is the angular part. Now Bohr [4] calculates the mass parameter B by the properties of irrotational flow for the nuclear matter with density ϱ_0 for a liquid drop of radius R_0 to be

$$B \sim \varrho_0 R_0^5. \tag{5}$$

On the other hand $d\Omega_4$ is the symmetric metric of S_4, with

$$\begin{aligned}
g_{\Phi\Phi} &= \frac{\mathcal{J}_1}{B}\sin^2\Theta\cos^2\psi + \frac{\mathcal{J}_2}{B}\sin^2\Theta\sin^2\psi + \frac{\mathcal{J}_3}{B}\cos^2\Theta, \\
g_{\Phi\Theta} &= \frac{1}{B}(\mathcal{J}_2 - \mathcal{J}_1)\sin\Theta\sin\psi\cos\psi, \\
g_{\Phi\psi} &= \frac{\mathcal{J}_3}{B}\cos\Theta, \\
g_{\Theta\Theta} &= \frac{\mathcal{J}_1}{B}\sin^2\psi + \frac{\mathcal{J}_2}{B}\cos^2\psi, \\
g_{\psi\psi} &= \frac{\mathcal{J}_3}{B}.
\end{aligned} \tag{6}$$

In general it can take various forms with respect to the choice of the angular part of the moments of inertia $\mathcal{J}_k$ [2],

$$\mathcal{J}_k = 4B\beta^2\sin^2\left(\gamma - k\frac{2\pi}{3}\right), \tag{7}$$

reflecting the constraints in the angles Ω_4 which generate the shapes of deformed nuclei. Moments of inertia are known to deviate from experimental data because of their rapid increase with respect to the deformation β [17]. The marked increase is generated the factor β^2, which is the radial part of all the $\mathcal{J}_k$ (7). This information shrinks in the line element (3), its second term shows that in Bohr geometry β^2 factorizes all the metric elements which refer to the angles.

On the other hand, it was already known from [18] that the rapid increase of the moments of inertia can be reduced by the inclusion of the pairing interaction which is absent in the Bohr Hamiltonian. Pairing is a manifestation of the finite number of particles that an atomic nucleus contain, in contrast with the liquid drop which demands a continuum. As Van Isacker and Chen had already noticed [9], in the Bohr Hamiltonian the shape variables are used *ab initio* ignoring the finite number of particles for an atomic nucleus. Bohr and Mottelson [18] explain that the liquid drop is a comparison of the collective motions of particles in atomic nuclei with the oscillations of an irrotational fluid with density ϱ_0, calculated by the properties of nuclear matter which is achieved by the extension of the mass number A at infinity. Yet, this comparison is quantitatively *implicit* in the Bohr Hamiltonian. The phenomenological deviations in the mass parameter could be generated by the geometrical absence of the infinite number of particles in the Bohr geometry [15]. The embedding of Bohr space in six dimensions introduces the reference to infinity and import new symmetries which should serve for the explicit comparison of the collective motions of finite number of nucleons with the oscillations of the irrotational fluid.

The geometry of the classical limit of the IBM is that of 5 + 1 dimensions, the six coordinates are in 1-1 correspondence with the coherent states of the $U(6)$. Bohr shape variables are obtained after the stereographic projection on the the five shape variables. The energy surfaces that correspond to the exact symmetry limits of $U(5)$, $O(6)$, $SU(3)$ and in the transitional regions between them, are very different from the Bohr Hamiltonian. They contain more complicated kinetic terms. In [19] the Bohr Hamiltonian was modified, by letting the mass parameter to depend on the deformation in the form

$$B = \frac{B_0}{(1 + a\beta^2)^2}, \tag{8}$$

with B_0 to be Bohr's mass parameter and a a constant parameter. Now Bohr's calculation for the moments of inertia (7) becomes

$$\mathcal{J}_k = 4\frac{B_0}{(1 + a\beta^2)^2}\beta^2 \sin^2\left(\gamma - k\frac{2\pi}{3}\right). \tag{9}$$

The β part is not rapidly increasing because of the factor $1/(1+a\beta^2)^2$ as discussed in [19]. Increasing values of the parameter a relax the increase of the moments of inertia with respect to β. The aim of [19] was to reveal new forms of kinetic terms resembling those of the IBM, but it was not actually resembling none of them even though the phenomenological deviation of the moments of inertia was corrected.

However, in [15] Bohr space is embedded in six dimensions by using the functional form of the dependence of the mass on the deformation as a conformal factor. A new metric is constructed

$$g_{ij} \to \tilde{g}_{ij} = \frac{1}{(1 + a\beta^2)^2} g_{ij}. \tag{10}$$

The mass parameter B, same as in (4), and the Laplace-Beltrami operator now gives

$$H = -\frac{\hbar^2}{2B}\left[\frac{(1 + a\beta^2)^5}{\beta^4}\frac{\partial}{\partial_\beta}\frac{\beta^4}{(1 + a\beta^2)^3}\frac{\partial}{\partial_\beta} - \frac{(1 + a\beta^2)^2}{\beta^2}\Lambda^2\right]. \tag{11}$$

By definition a conformal transformation preserves the angles thus the $SO(5)$ invariance is preserved. The conformal factor does not affect B, the scalar factor $B \sim \varrho_0 R_0^5$ remains. The $SO(5)$ invariance does not change the angles and therefore the importation of the conformal factor in the Bohr model should not violate irrotational flow.

In [15] it is proved that the Bohr metric with the conformal factor emerges from the stereographic projection of a sphere with five angles S_5, on to the tangent plane which is the Bohr space. The stereographic projection is equivalent with the conformal transformation of the Bohr line element. This is seen by the projected radial variable which is

$$\tilde{\beta} = \frac{\beta}{1 + a\beta^2}. \tag{12}$$

Now, Bohr's calculation for the moments of inertia gives

$$\mathcal{J}_k = 4B\tilde{\beta}^2 \sin^2\left(\gamma - k\frac{2\pi}{3}\right) = 4B\frac{\beta^2}{(1 + a\beta^2)^2}\sin^2\left(\gamma - k\frac{2\pi}{3}\right), \tag{13}$$

which is equivalent with the one emerged from deformation dependent mass (9). This equivalence is also expressed in terms of a mass parameter

$$B = \frac{B(0)}{(1 + a\beta^2)^2}, \tag{14}$$

as Van Isacker and Heyde used it recently in [3]. $B(0)$ is the mass on zero deformation which in irrotational flow is $B(0) \sim \varrho_0 R_0^5$, i.e Bohr's prediction.

As analyzed in [15] the reduction of the moments of inertia with this approach reveals new symmetries principally absent from the Bohr model. The symmetry group of S_5 is the $O(6)$ and it is proved [15] that corresponds to the $O(6)$ limit of the IBM. It deserves to be noticed that the $O(6)$ limit of the IBM contains the pairing interaction [1]. Therefore, this manifestation of $O(6)$ should serve for the explicit comparison of the nuclear collective motions that generate the $O(6)$ coherent states with the oscillations of the nuclear fluid, which is located for $a = 0$.

The radius of the S_5 is $\frac{1}{\sqrt{4a}}$ revealing the physical content of the parameter a, it is the $\frac{1}{4}$ of the inverse radius of the S_5. For $a = 0$ the metric (10) gives the Bohr metric (3). This is the limit of infinite radius of the S_5. Bohr space is located at the boundary at infinity and it corresponds to the limit of the large N of the IBM [15]. Finite values of a correspond to a constraint on the number operator.

The projection of the S_5 is equivalent with the importation of a conformal factor in Bohr's space. In turn the absence of the conformal factor reflects the absence of $O(6)$ fingerprints. Parameter a controls the presence of the conformal factor. For $a = 0$, $E(5)$ symmetry emerges as the contraction of $O(6)$ at infinity. In other words, the hamiltonian of $E(5)$ appears as a certain limit of the Hamiltonian (11), for $a = 0$ it gives the Casimir invariant of $E(5)$.

The Hamiltonian (11) corresponds to the $U(5) - O(6)$ limit of the IBM [15], and reveals the $E(5)$ as the limit of (11) for $a = 0$. In addition, $a = 0$ corresponds to the limit of the large N of the IBM. This is another argument towards the identification of the $E(5)$ as the critical point for the 2nd order phase transition of the IBM. It satisfies the Gibbs criterion as is revealed in the limit of the large N of the $U(5) - O(6)$ transitional region of the IBM.

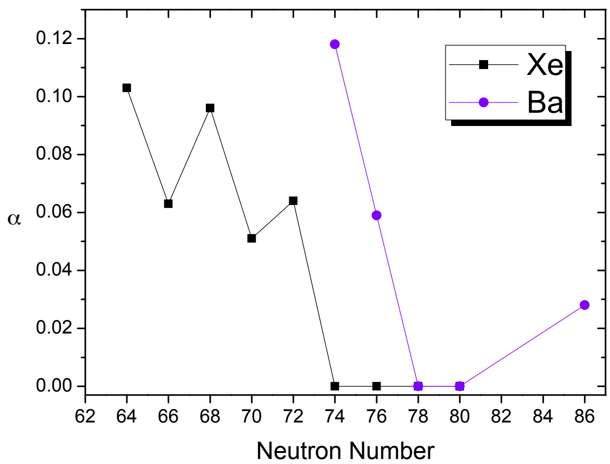

Figure 1: Values of a for Xe and Ba isotopes, data taken from [19].

The $E(5)$ candidates will be the nuclei which are fitted to the value of $a = 0$ but for the solutions of the (11). Predictions of $E(5)$ manifestations in atomic nuclei which lean on the spectrum of the initial Bohr Hamiltonian with the infinite square well [5], which is the H_B (4) with an extra parameter, are rather incomplete. Bohr's Hamiltonian (4) is different from the Hamiltonian (11) and their phenomenology should be different.

Parameter a controls the presence of the conformal factor. In [19], a-values are obtained by their rms fitting to the experimental data for the energy spectrum of a plethora of γ-unstable nuclei. A close examination of its available values draws attention to the nuclei for which $a = 0$. Fig. 1 displays the a-values for the Xe and Ba isotopes. Parameter a is not zero for a single nucleus in the series of Xe, or in that of the Ba isotopes. $a = 0$ corresponds to the series of ^{128} Xe, ^{130} Xe, ^{132} Xe, ^{134} Xe as well as in those of ^{134} Ba and ^{136} Ba. This remark should not be received as a proposal for $E(5)$ candidates. In [19] for

$a = 0$ the Hamiltonian of $E(5)$ [5] is not obtained, Davidson term β^2 survives which nevertheless has been proposed to characterize the transitional region [16]. The present approach implies that the manifestation of $E(5)$ in atomic nuclei, needs additional experimental measures than those obtained by the initial Bohr Hamiltonian. A geometrical limit of the IBM containing the parameter a should illustrate its appearance in nuclear structure.

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