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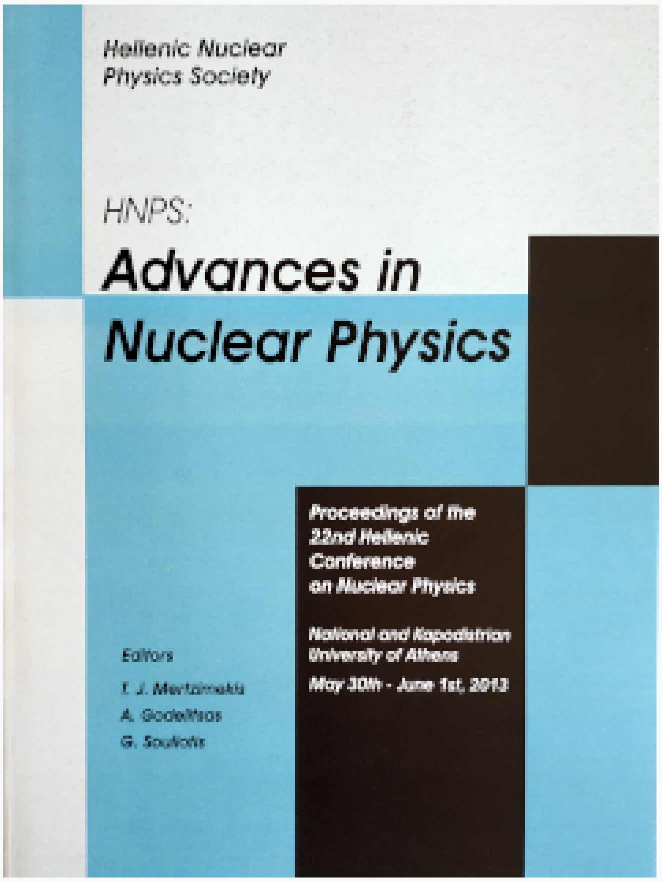

Parafermionic behavior of Bose-Einstein condensates

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Parafermionic behavior of Bose-Einstein condensates

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Abstract

Bright solitons of ^{7}Li atoms in a quasi one-dimensional optical trap, formed in a stable Bose–Einstein condensate in which the interactions have been magnetically tuned from repulsive to attractive, have been seen to exhibit repulsive interactions among them when set in motion by offsetting the optical potential. Solving first the Gross–Pitaevskii equation for the special conditions corresponding to the experiment, we show then that this system can be described in terms of generalized parafermionic oscillators, the order of the parafermions being related to the strength of the interaction among the atoms and being a measure of the bosonic behavior vs. the fermionic behavior of the system.

Keywords: Parafermions, Bose-Einstein condensates

1. Introduction

Elementary particles can be either fermions, for which each quantum state can be occupied by only one particle because of the Pauli principle, or bosons, for which each quantum state can be occupied by any number of particles up to infinity. However, composite bodies can present intermediate statistics [1], behaving as parafermions of order p, for which each quantum state can be occupied by up to p particles. Clearly fermions are parafermions of order one, while bosons are parafermions of infinite order.

Parafermions can be conveniently described in the framework of generalized deformed oscillators [2–4], using a formalism similar to that of the usual harmonic oscillator in terms of creation and annihilation operators. Parafermionic creation and annihilation operators have been used to express the Hamiltonian of the system. The parameters of the responding energy have been fitted on experimental results.

Stable bright solitons have been recently constructed in Rice University, [5–8] in a stable Bose–Einstein condensate of ^{7}Li atoms, cooled down at almost zero Kelvin temperature and trapped in a cylindrical, magneto-optical trap. The interaction among the lithium atoms has been tuned from repulsive to attractive through a strong magnetic field. The solitons were created when the interaction among the lithium atoms was attractive. The motion of the solitons, triggered by offsetting the optical potential, revealed repulsive interactions among neighboring solitons, although the interactions among the lithium atoms were attractive. In other words, solitons consisting of bosons with attractive interactions among themselves, exhibit a fermion-like behavior, repelling each other.

2. Parafermionic Oscillators

The mathematical framework of the theory is based on the definition of deformed annihilation and creation operators $a, a^\dagger$ which satisfy the following relations [2, 3]

$$[a, N] = a, \qquad \left[a^+, N\right] = -a^+, \tag{1}$$

$$a^+a = \Phi(N) = [N], \qquad aa^+ = \Phi(N+1) = [N+1], \tag{2}$$

where $\Phi(x)$ is a positive analytic function with $\Phi(0) = 0$, N is the number operator, and the notation $[x] = \Phi(x)$ is used for brevity.

The physical content of the structure function is revealed by computing the commutation and anticommutation relations that follow from the above definitions

$$[a, a^\dagger] = \Phi(N+1) - \Phi(N), \tag{3}$$

$$\{a, a^\dagger\} = \Phi(N+1) + \Phi(N). \tag{4}$$

We note that these are not the usual bosonic commutation relations and fermionic anticommutation relations. Therefore the structure function is characteristic of the statistics of the deformation scheme.

The appropriates structure function for the condensate is [9]

$$\Phi(N) = N(p+1-N)(\lambda + \mu N + \nu N^2 + \rho N^3 + \ldots), \tag{5}$$

where λ, μ, ν, ρ are real constants satisfying the conditions

$$\lambda + \mu N + \nu N^2 + \rho N^3 + \ldots > 0 \quad \text{if } N \in \{1, 2, ...\}, \tag{6}$$

while p is the order of the parafermions, which reveals the number of particles that can occupy the same quantum state [4]. If $p = 1$ the particles are fermions, while if $p \to \infty$ the particles are bosons. The order p of the parafermions is a measure of the *bosonic* or *fermionic* behavior of a system. The larger the order p is, the more bosonic the behavior of the system is.

3. The energy of the Bose-Einstein condensate

We consider a Bose-like Hamiltonian [9]

$$H_B = \frac{1}{2}\{a, a^\dagger\} = \frac{1}{2}(aa^\dagger + a^\dagger a). \tag{7}$$

Although this Hamiltonian mimics the Hamiltonian of a system of bosons, it is not identical to it, since the operators a, $a^\dagger$ are the deformed ones.

The eigenvalues of the Bose-like Hamiltonian can be calculated in the Fock space described above using Eq. (4)

$$E_B(N) = \frac{1}{2}(\Phi(N) + \Phi(N+1)). \tag{8}$$

Using Eq. (5), and keeping terms up to N^4, the energy E_B of the generalized parafermionic oscillator as a function of the number of particles N is

$$\begin{aligned} E_B(N) = \frac{1}{2}((\lambda + \mu + \nu)p + ((2\lambda + 2\mu + 3\nu)p - \mu - \nu)N \\ + ((2\mu + 3\nu)p - 2\lambda - \mu - 3\nu)N^2 \\ + (2\nu p - 2\mu - 2\nu)N^3 - 2\nu N^4). \end{aligned} \tag{9}$$

The above has been the parafermionic way to obtain the energy. To the same purpose the majority of scientists use the Gross-Pitaevskii equation

$$i\hbar \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = (-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + g|\Phi(\mathbf{r}, t)|^2)\Phi(\mathbf{r}, t), \tag{10}$$

where the coupling constant g is due to the interaction potential among the atoms and has negative values, $\Phi(\mathbf{r}, t)$ is the macroscopic wave function of the condensate and m is the mass of one atom. The V_{ext} is the potential of the magneto optical trap where the condensate is formed. Various experiments have used different shapes of traps. For a cylindrical trap only the axial z direction is of interest. The axial part of the wave function is

$$U(z, t) = A\ \eta\ \text{sech}[\eta(z - vt)]\ e^{i(kz - \omega t)}, \tag{11}$$

where v is the condensate velocity, k the wavenumber, ω the frequency, η and A parameters relevant to the scattering length of the atoms.

The energy can be calculated from [10]

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} (\frac{\hbar^2}{m}|\frac{\partial U}{\partial z}|^2 + g_{1D}|U|^4)dz. \tag{12}$$

After calculations the energy is found to be

$$E = \frac{\hbar^2 k^2}{2m}N + \frac{1}{12A^2}\left(g_{1D} + \frac{\hbar^2}{2mA^2}\right)N^3. \tag{13}$$

The first term of the above equation contains the kinetic energy of the free particle, $K = \frac{\hbar^2 k^2}{2m}$. The energy of the BEC is a nonlinear function of N, due to the second term in Eq. (13). Therefore the lithium atoms do not behave like ideal bosons. Moreover, the nonlinear term is proportional to the coupling constant g_{1D}, which is a measure of the strength of the interatomic attraction. The consequence of this proportionality is that the stronger the interaction among the atoms is, the largest are the deviations from the bosonic behavior.

One can obtain the order parameter by comparing equations (8) and (13). In the large N limit

$$p^2 = \frac{K}{\mu}, \tag{14}$$

and

$$E = c_1 N + c_3 N^3, \qquad c_1 = K, \qquad c_3 = -\mu, \tag{15}$$

The term proportional to N corresponds to the pure bosonic behavior, while the N^3 term results from additional interactions perturbing the bosonic system and leads to parafermionic behavior. The absolute value of the ratio of the two coefficients

$$\left|\frac{c_1}{c_3}\right| = p^2 \tag{16}$$

is therefore a measure of the bosonic vs. fermionic behavior. The degree of bosonic vs. fermionic behavior of the system depends both on the kind of the atoms used and on the confinement conditions.

4. Conclusion

The main findings of this paper are summarized here.

1) The energy as a function of the number of particles of an ultra cold gas provides information about the fermionic or bosonic behavior of the gas. Specifically, in an ultra cold gas the linear term is evidence of bosonic behavior, while the nonlinear terms indicate a fermionic character.

2) The statistical behavior of non-elementary particles depends on the kind of the particles and on the confinement conditions. Therefore, atoms manifesting bosonic behavior in dilute gases, can form condensates exhibiting a fermionic-like character when confined in one dimension.

3) Interacting bosons within a Bose–Einstein condensate confined in one dimension can be described as parafermions, the order of the parafermions being related to the strength of the attractive interaction among the atoms. No such representation is possible in the case of repulsive interactions among the atoms.

4) The order of these parafermions turns out to be a measure of the bosonic behavior vs. the fermionic behavior.

5) As a result, bright solitons within a Bose–Einstein condensate in one dimension exhibit repulsion among themselves, as a consequence of their parafermionic nature.

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