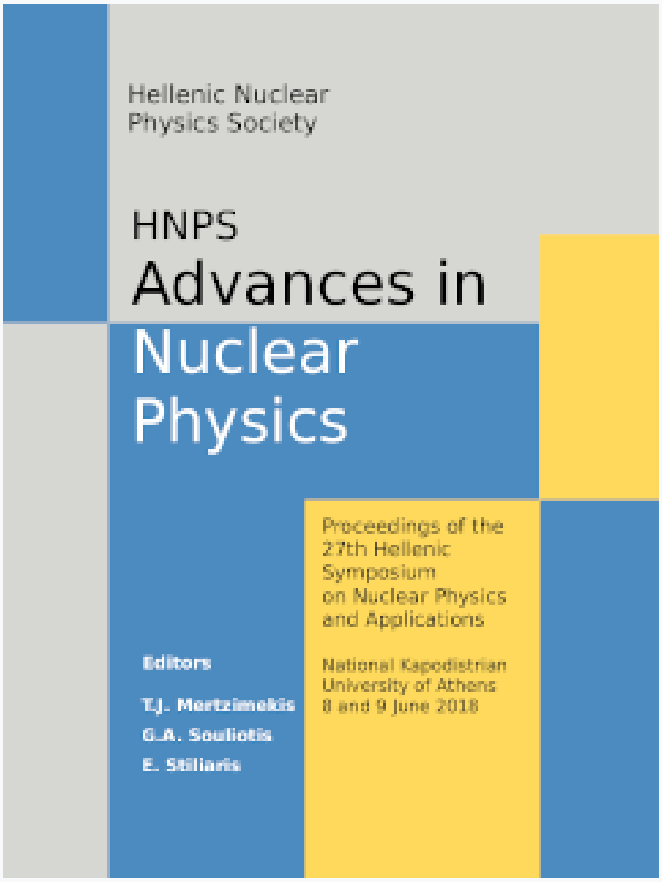
HNPS Advances in Nuclear Physics

Vol 26 (2018)

HNPS2018



KIDS Energy Density Functional and Mass–Radius Relation of Neutron Stars

G. Ahn, P. Papakonstantinou

doi: [10.12681/hnps.1805](https://doi.org/10.12681/hnps.1805)

KIDS Energy Density Functional and Mass – Radius Relation of Neutron Stars

G. Ahn[1], P. Papakonstantinou[2,*]

[1] *National and Kapodistrian University of Athens, Department of Physics, Nuclear and Particle Physics Section, GR 15784, Athens, Greece*
[2] *Rare Isotope Science Project, Institute for Basic Science, Daejeon 34047, South Korea*

Abstract Many efforts are made to determine the nuclear equation of state which governs the properties and evolution of neutron stars. Especially important is to constrain the parameters of the nuclear symmetry energy. In those efforts, nuclear energy density functional (EDF) theory has been a very useful tool, as it provides a unified framework for the description both of nuclei, which can be studied on Earth, and of infinite matter and its nuclear equation of state, which is a necessary input in the modelling of neutron stars. In the present study, a new nuclear EDF, the KIDS functional, is explored with a focus on the nuclear symmetry energy. The form of the functional allows us to vary at will the poorly constrained high-order derivatives of the symmetry energy and examine how the maximum possible mass of a neutron star is affected. Some tentative constraints on the skewness are presented, which will help guide further refinements. It is noteworthy that the pressure of neutron-rich matter is found strongly affected by skewness variations, both at low and high densities.

Keywords KIDS functional, Symmetry Energy, Equation of State, Mass-Radius Relation, Neutron Stars

---

INTRODUCTION

Much intense effort is devoted to determining the nuclear equation of state which governs the properties and evolution of neutron stars. Especially important is to constrain the parameters of the nuclear symmetry energy. In the process, nuclear energy density functional (EDF) theory [1] has been a very useful tool, as it provides a unified framework for the description both of nuclei, which can be studied on Earth, and of infinite matter and its nuclear equation of state. Most functionals that are used in describing the nuclear matter are fitted to experimental data and in particular nuclear masses, charge radii, and giant resonances. This leads to a fairly reliable description of nuclear matter at and around the saturation density. However, extrapolations to high and low densities must be carried out with care [2]. In addition, overfitting to ordinary nuclei may result in decreased predictive power in exotic, neutron-rich systems.

An important component of the equation of state is the symmetry energy, which is usually characterized by its value at saturation density, as well as its derivatives' values at saturation density. This quantity affects the structure of stable and exotic nuclei, and also affects neutron stars' properties [3]. Dedicated efforts over the years have led to ever-

---

* Corresponding author, email: ppapakon@ibs.re.kr

narrowing constraints for the symmetry energy value and its slope [4]. Explorations at high density through heavy-ion collisions as well as astronomical observations are expected to illuminate the role of higher-order derivatives, which are not often considered and their values are not well constrained.

A method which allows us to choose at will the values of undetermined quantities and apply them in describing nuclei and neutron stars has been proposed in the framework of the KIDS functional [5, 6]. It will allow us to study the sensitivity of calculated observables to each of the above unconstrained quantities and thus systematically narrow down their values. Such a methodology would be tedious if not impossible in the context of usual functionals, which have a limiting analytical form (cf Skyrme functionals) and in addition require each time a fit on data of finite nuclei. By contrast, the form of the KIDS functional is optimal for varying any quantity of interest independently of the others. We explore this possibility to study the effect of the symmetry energy's curvature and skewness on the neutron star equation of state.

The present manuscript is organized as follows. In the the next two sections we present the optimal Ansatz of the KIDS functional, its analytical relations to the symmetry energy, and the formalism that relates the symmetry energy with the neutron-star properties. For the subsequent studies, we select as baseline an initial KIDS parameterization (KIDS-ad-2 [5]) which corresponds to a realistic equation of state (Akmal-Pandharipande-Ravenhall [7] for pure neutron matter) and has been successfully applied in the description of magic nuclei [6]. We then explore 10 equations of state which are nearly identical (in terms of saturation properties) to ad-2 and identical to each other except for the curvature and skewness of the symmetry energy. We find deviations for the pressure both at low and high densities and in the description of neutron stars indicating the significance of these parameters.

KIDS FUNCTIONAL AND SYMMETRY ENERGY

It is known from Brueckner's theory for homogenous matter and from effective field theories for dilute Fermi systems that the Fermi momentum (κ_F), which is proportional to the cubic root of the density, is a fundamental quantity and indispensable variable in the description of fermionic systems [8, 9, 10]. The recently proposed KIDS (Korea – IBS – Daegu – Sungkyunkwan) energy density functional therefore uses relevant powers of the cubic root of the density $\left(\rho^{1/3}\right)$ [5]. In particular, the energy per particle in the context of the KIDS functional is:

$$\varepsilon(\rho,\delta) = \frac{E(\rho,\delta)}{A} = T(\rho,\delta) + \sum_{i=0}^{3} c_i(\delta)\,\rho^{1+i/3},$$

where $T(\rho,\delta)$ is the free Fermi gas kinetic energy, $c_i(\delta)$ the KIDS parameters and $\delta = \frac{\rho_n-\rho_p}{\rho}$ the asymmetry of nuclear matter.

There are two special cases that we are particularly interested in modeling. The first one is when the asymmetry is zero, which means that the number of protons in the matter is equal to the number of neutrons. In this kind of matter, the so-called Symmetric Nuclear Matter (SNM), the following apply:

$$\rho_p = \rho_n, \delta = 0$$

and the energy per particle reduces to:

$$\frac{E_{SNM}}{A} = \frac{3\hbar^2}{10M}\left(\frac{3\pi^2}{2}\right)^{2/3} \rho^{2/3} + \sum_{i \geq 0} c_i(0)\, \rho^{1+i/3}.$$

The second case of interest is when there are no protons in the matter and only neutrons exist. In this kind of matter, which is called Pure Neutron Matter (PNM), the following apply:

$$\rho_p = 0, \rho_n = \rho, \delta = 1$$

and the energy per particle reduces to:

$$\frac{E_{PNM}}{A} = \frac{3\hbar^2}{10M}(3\pi^2)^{2/3} \rho^{2/3} + \sum_{i \geq 0} c_i(1)\, \rho^{1+i/3}.$$

Interpolations between the two cases can be done using the quadratic approximation,

$$c_i(\delta) \approx c_i(0) + [c_i(1) - c_i(0)]\delta^2.$$

They can also be done through the formalism of Skyrme functionals, particularly useful for finite nuclei [6]. From the energy-per-particle equation it is possible to extract analytical relations for the important quantities that characterize the symmetric nuclear matter. Those are the saturation density (ρ_0), the energy per particle at saturation density (ε_0) and the incompressiblity (K_0).

An important quantity is the Symmetry energy (E_{sym}), which is defined as [2]:

$$E_{sym}(\rho) \equiv \frac{1}{2}\left(\frac{\partial^2 E(\rho,\delta)}{\partial \delta^2}\right)_{\delta=0} \simeq E(\rho,1) - E(\rho,0).$$

It describes the static response of the nucleus to the neutron-proton asymmetry, it affects many nuclear phenomena, such as the stability of neutron-rich nuclei and ultimately, the mass-radius relation of neutron stars.

We are interested in E_{sym} and its derivatives with respect to the density at the saturation point. In particular, the quantities of interest are the symmetry energy $\left(J = E_{sym}(\rho_0)\right)$, the slope of the symmetry energy (L), the curvature of the symmetry energy $\left(K_{sym}\right)$ and the skewness (Q_{sym}), all at saturation density. In the context of the KIDS functional they are given by*:

$$J \equiv S_0 \equiv E_{sym}(\rho_0) = \frac{\hbar^2}{6M}\left(\frac{3\pi^2}{2}\right)^{2/3} \rho_0^{2/3} - \sum_{i=0}^{3}\left(c_i(0) - c_i(1)\right)\rho_0^{1+i/3}$$

$$L = \frac{\hbar^2}{3M}\left(\frac{3\pi^2}{2}\right)^{2/3} \rho_0^{2/3} - \sum_{i=0}^{3}(3+i)\left(c_i(0) - c_i(1)\right)\rho_0^{1+i/3}$$

---

* Here we adopt the simplified expression for the kinetic term from Ref. [2]. A somewhat different one was used in Ref. [4], leading to somewhat different values for the reported symmetry-energy parameters. The consequences are minor in the present context.

$$K_{sym} = -\frac{\hbar^2}{3M}\left(\frac{3\pi^2}{2}\right)^{2/3}\rho_0^{2/3} - \sum_{i=0}^{3}(3+i)\,i\,\big(c_i(0) - c_i(1)\big)\rho_0^{1+i/3}$$

$$Q_{sym} = \frac{4\hbar^2}{3M}\left(\frac{3\pi^2}{2}\right)^{2/3}\rho_0^{2/3} - \sum_{i=0}^{3}(3+i)\,i(3-i)\,\big(c_i(0) - c_i(1)\big)\rho_0^{1+i/3}.$$

The above quantities are very important and will help us characterize the nuclear matter. The optimal number of KIDS parameters and the above analytical expressions will allow us to determine a set of KIDS functional parameters for any combination of (J, L, K_{sym}, Q_{sym}) set we wish to examine.

Experimental observations suggest that the symmetry energy at saturation density should be $J = (30 - 35)$ MeV, the slope of the symmetry energy at saturation density should be $L(\rho_0) = (40 - 76)$ MeV and the curvature of the symmetry energy at saturation density should be $K_{sym}(\rho_0) = (-100 \pm 100)$ MeV. While the skewness is not well constrained yet, a strong correlation with the slope seems to exist: $Q_{sym}(\rho_0) = -6.443L(\rho_0) + (708.74 \pm 118.14)$ MeV [11, 12, 13, 14].

EQUATION OF STATE AND TOV EQUATIONS

The equation of state (EoS) is the equation that gives the energy per particle (or the energy density) as a function of the density $(\varepsilon(\rho))$. We will focus on the EoS of stellar matter, namely charge-neutral β-stable matter, which is encountered in the core of neutron stars and on the EoS of dilute ($\rho \leq 0.2$ fm^{-3}) neutron matter, which is encountered in the inner crust of neutron stars. Stellar matter consists mostly of neutrons, but a non-zero proton and lepton fraction is present.

In order to find the mass-radius relation of neutron stars, which is an essential step in understanding neutron stars, the Tolman-Oppenheimer-Volkoff (TOV) equations must be solved. For non-rotating neutron stars, these equations are:

$$\frac{dP}{dr} = -G\frac{M(r) + 4\pi r^3\frac{P(r)}{c^2}}{r\big(rc^2 - 2GM(r)\big)}$$

$$\frac{dM}{dr} = \frac{4\pi\varepsilon(r)r^2}{c^2}$$

where $P(r)$ the pressure and $\varepsilon(r)$ the energy density of β-stable matter.

The TOV equations require an EoS that describes the stellar matter. From the EoS of neutron matter, it is possible to find the EoS of the stellar matter using charge neutrality and equalities involving chemical potentials of the nucleons and leptons [15].

Before we proceed with our results, we must clarify that this is work in progress and our present results should be considered preliminary and indicative. In particular, in the present study, and unlike Ref. [5], 1) the low-density regime (spinodal region, clusterization and the core-crust transition) has not been adequately modeled and 2) the causality condition has been applied to PNM, not beta-stable matter. Given these shortcomings, we expect that extracted values for the maximum mass of a neutron star are the most robust of our current results.

FINDING THE MASS-RADIUS RELATION OF NEUTRON STARS

In Ref [5] a set of parameters for the KIDS functional was determined based on generally adopted properties of SNM ($\rho_0 = 0.16\ \mathrm{fm}^{-3}, \varepsilon_0 = -16\mathrm{MeV}, K_0 = 240\mathrm{MeV}$) and on a fit of the PNM EoS to the Akmal-Pandharipande-Ravenhall EoS [7]. The parameters for that set, labeled ad-2, are shown in the following table:

	$c_0(\delta)$ (MeV fm^3)	$c_1(\delta)$ (MeV fm^4)	$c_2(\delta)$ (MeV fm^5)	$c_3(\delta)$ (MeV fm^6)
SNM (δ=0)	-664.52	763.55	40.13	0.00
PNM (δ=1)	-411.13	1007.78	-1354.64	956.47

Table 1. The ad-2 parameters of the KIDS functional [5].

We use the above parameter set as baseline and henceforth fix the SNM parameters to the same values and the symmetry energy parameters J and L to the rounded $J = 33$ MeV, $L = 50$ MeV values (see also Table 3). For the purposes of this study we then vary the values of the curvature (2 values) and skewness (5 values): $K_{sym} = -160$ MeV (rounded ad-2 value) and 0 MeV; $Q_{sym} = (-200, 0, 400, 600, 1000)$ MeV. New sets of values of $c_i(\delta)$ are readily obtained through the available analytical relations and tabulated below.

K_{sym} (MeV)	Q_{sym} (MeV)	$c_0(1)$	$c_1(1)$	$c_2(1)$	$c_3(1)$
-160	-200	382.50	-3395.77	6801.91	-4077.37
-160	0	174.16	-2244.51	4681.28	-2775.29
-160	400	-242.50	58.01	440.00	-171.12
-160	**600**	**-450.84**	**1209.27**	**-1680.64**	**1130.96**
-160	**1000**	**-867.50**	**3511.79**	**-5921.92**	**3735.13**
0	-200	1382.50	-8000.81	13587.96	-7202.37
0	0	1174.16	-6849.55	11467.32	-5900.29
0	400	757.50	-4547.03	7226.04	-3296.12
0	600	549.16	-3395.77	5105.40	-1994.04
0	**1000**	**132.50**	**-1093.25**	**864.13**	**610.13**

Table 2. The parameters of the KIDS functional that were found for $J = 33$ MeV, $L = 50$ MeV and for K_{sym} and Q_{sym} that can be seen in the table above. The units of c_i are MeV $\cdot$ fm^{3+i}.

Using the above parameters we find the energy per particle for PNM, which is shown in Fig.1, normalized to the free Fermi energy. Although the description of the saturation region is practically the same for all sets, away from the saturation point large deviations are observed. This shows the significance of the high-order parameters of the symmetry energy, namely its curvature and skewness, at both low and high densities. Next we determine the respective EOSs of stellar matter. Its pressure for the 10 parameter sets is shown in Fig.2. We observe that the pressure for some sets becomes negative already at densities relevant for neutron stars. The varying range of the spinodal region at subsaturation densities for given K_{sym} is remarkable, given that the only difference among the displayed equations of state in each graph is the value of Q_{sym}.

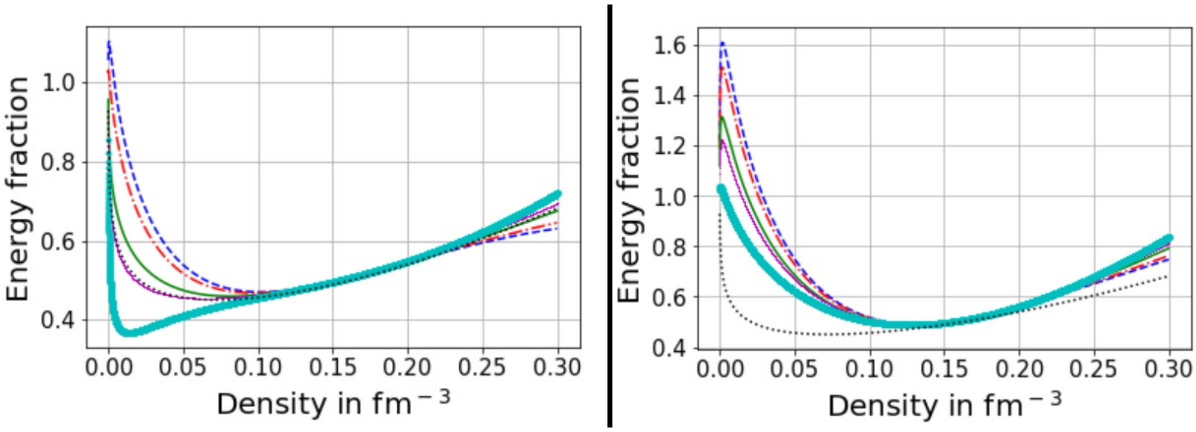


Figure 1. Left: Energy per particle graph for PNM, normalized to the free Fermi energy, versus density for $J = 33$ MeV, $L = 50$ MeV, $K_{sym} = -160$ MeV and various values of Q_{sym}. Blue dashed line corresponds to $Q_{sym} = -200$ MeV, red dash-dotted line to $Q_{sym} = 0$ MeV, green solid line to $Q_{sym} = 400$ MeV, magenta pixelled line to $Q_{sym} = 600$ MeV, cyan thick line to $Q_{sym} = 1000$ MeV and black dotted line to ad-2 parameters. Right: Same as left, but with $K_{sym} = 0$ MeV.

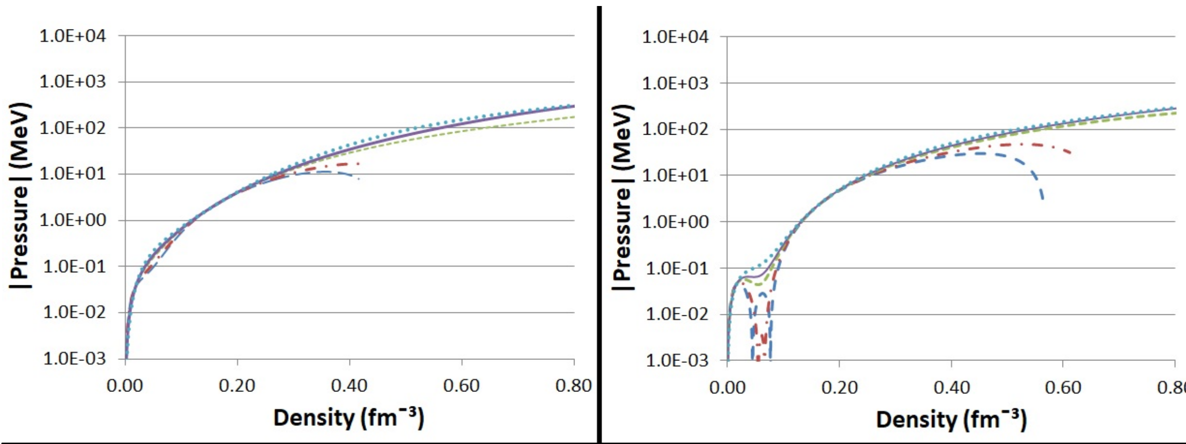


Figure 2. Left: Pressure versus density for $J = 33$ MeV, $L = 50$ MeV, $K_{sym} = -160$ MeV and various values Q_{sym}. Blue dashed line corresponds to $Q_{sym} = -200$ MeV, red dash-dotted line to $Q_{sym} = 0$ MeV, green short-dashed line to $Q_{sym} = 400$ MeV, purple solid line to $Q_{sym} = 600$ MeV and cyan dotted line to $Q_{sym} = 1000$ MeV. Right: Same as left, but for $K_{sym} = 0$.

Only three of the sets could satisfy fundamental physical (at present, also technical) constraints which had to do with the stability behavior of the pressure, namely positive values for the pressure and its gradient especially at high densities. Those three sets of parameters are indicated in boldface in Table 2 and henceforth we will refer to them as (I), (II) and (III).

The corresponding sets of values for the symmetry energy and its derivatives are summarized and compared to those of ad-2 in Table 3. Note that set (I) is almost identical to ad-2 by construction.

	J (MeV)	L (MeV)	K_{sym} (MeV)	Q_{sym} (MeV)
(I)	33.00	50.00	-160.00	600.00
(II)	33.00	50.00	-160.00	1000.00
(III)	33.00	50.00	0.00	1000.00
ad-2	32.76	49.11	-156.69	586.29

Table 3. The symmetry-energy parameters for which the discussed stability requirements are satisfied (indicated with boldface in Table 2), compared with those of ad-2.

Using the EoSs that correspond to the sets of Table 3, we solved the TOV equations to find the respective mass-radius relations. At this stage, a causality check is necessary, so that the mass-radius relations found are not superluminal. For this check, the critical maximum density for which the speed of sound becomes equal to the speed of light at vacuum was found and the graph was cut from that point onwards. (As already mentioned, at present the check was performed on PNM rather than stellar matter, which renders the results approximate. This is also why the present results for ad-2 deviate somewhat from the more accurate ones of Ref. [5].) The critical density can be seen in Table 4, along with the maximum predicted mass of the neutron star and a reference radius of a neutron star with a particular mass ($1.4\ M_{\odot}$).

	Max Mass($M_{\odot}$)	$R_{1.4}$(km)	ρ_{max}(fm^{-3})
(I)	2.05	11.29	0.877
(II)	1.92	11.70	0.615
(III)	1.96	12.07	0.632
ad-2	2.06	11.27	0.906

Table 4. Preliminary results (see text for current technical limitations) for the maximum predicted mass of a neutron star, a reference radius of a neutron star with a particular mass and the maximum density for which causality criterion is satisfied. The labelling is the same as Table 3.

The latest observations have confirmed that a neutron star is able to have a mass greater than $2\ M_{\odot}$. Out of the three sets suggested above, only one set (I) is able to yield a maximum mass of a neutron star greater than $2\ M_{\odot}$ [16], at least within the technical limitations of the present study.

CONCLUSIONS

The target of this study was to investigate the symmetry energy in the context of the KIDS functional, using the symmetry energy and its derivatives, as well as some of the latest astronomical observations. An important result is that the skewness (Q_{sym}) of the symmetry energy significantly affects the pressure and the mass-radius relation of the neutron stars, if varied within the full range of current uncertainties. For example, we found that given $(\mathrm{J}, \mathrm{L}, \mathrm{K}_{\mathrm{sym}}) = (33, 50, -160)$ MeV, the skewness $(\mathrm{Q}_{\mathrm{sym}})$ should be at least 400 MeV, while for other $\mathrm{K}_{\mathrm{sym}}$, the resulting allowed $\mathrm{Q}_{\mathrm{sym}}$ are different. Given that J and L were kept constant, the present study hints at the possibility to confidently constrain the values of $\mathrm{K}_{\mathrm{sym}}$, $\mathrm{Q}_{\mathrm{sym}}$ with the help of astronomical observations.

Acknowledgements

We are thankful to Prof. E. Mavrommatis for useful comments on the manuscript. Work supported by the Rare Isotope Science Project of the Institute for Basic Science funded by Ministry of Science, ICT and Future Planning and the National Research Foundation (NRF) of Korea (2013M7A1A1075764).

References

[1] M. Bender et al., Rev. Mod. Phys. **75**, 121 (2003)
[2] M. Dutra et al., Phys. Rev. C **85**, 035201 (2012)
[3] W. Nazarewicz et al., Eur. Phys. J. A **50**, 20 (2014)
[4] B.-A. Li, Nucl. Phys. News, **27**, 4 (2017)
[5] P. Papakonstantinou et al., Phys. Rev. C **97**, 014312 (2018)
[6] H. Gil et al., arXiv: 1805.11321
[7] A. Akmal et al., Phys. Rev. C **58**, 1804-1828 (1998)
[8] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (Dover Publications, New York, 1971)
[9] H. W. Hammer and R. J. Furnstahl, Nucl. Phys. A **678**, 277-294 (2000)
[10] N. Kaiser et al., Nucl. Phys. A **697**, 255-276 (2002)
[11] J. R. Stone and P.-G. Reinhard, Prog. Part. Nucl. Phys. **58**, 587 (2007)
[12] L. W. Chen et al., Phys. Rev. C **82**, 024321 (2010)
[13] J. Margueron et al., Phys. Rev. C **97**, 025805 (2018)
[14] I. Tews et al., Astrophys. J. **848**, 105 (2017)
[15] P. Haensel, A. Y. Potekhin and D. G. Yakovlev, *Neutron Stars 1: Equation of State and Structure* (Springer, 2007)
[16] J. Antoniadis et al., Science **340**, 448 (2013)